NONLINEAR POTENTIAL THEORY AND QUASILINEAR EQUATIONS WITH MEASURE DATA

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THÉORIE NON LINÉAIRE DU POTENTIEL ET ÉQUATIONS QUASILINÉAIRES AVEC DONNÉES MESURES

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THÉORIE NON LINÉAIRE DU POTENTIEL ET EQUATIONS QUASILINÉAIRES AVEC DONNÉES MESURES

Résumé

Cette thèse concerne l'existence et la régularité de solutions d'équations non-linéaires elliptiques, d'équations paraboliques et d'équations de Hesse avec mesures, et les critères de l'existence de solutions grandes d'équations elliptiques et paraboliques non-linéaires.

Liste de publications

1. Avec M. F. Bidaut-Véron, L. Véron ; *Quasilinear Lane-Emden equations with absorption and measure data,* Journal des Mathématiques Pures et Appliquées, **102**, 315-337 (2014).

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3 Avec L. Véron ; *Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption,* 17 pages, soumis, arXiv :1308.2956.

4 Avec M. F. Bidaut-Véron ; *Stability properties for quasilinear parabolic equations with measure data,* 29 pages, à apparaître dans Journal of European Mathematical Society, arXiv :1409.1518.

5 Avec M. F. Bidaut-Véron ; *Evolution equations of p-Laplace type with absorption or source terms and measure data*, 21 pages, à apparaître dans Communications in Contemporary Mathematics, arXiv :1409.1520.

6 *Potential estimates and quasilinear parabolic equations with measure data,* 118 pages, arXiv :1405.2587v1.

7 Avec L. Véron ; *Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain,* 29 pages, soumis, arXiv :1406.3850.

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Mots clés : équations quasi-linéaire elliptiques, équations quasi-linéaires paraboliques ; solutions renormalisée ; solutions maximales ; solutions grandes ; potentiel de Wolff ; potentiel de Riesz ; potentiel de Bessel ; potentiel maximal ; Noyau de la chaleur ; noyau de Bessel parabolique ; Mesures de Radon ; capacités de Lorentz-Bessel ; capacités de Bessel ; capacités de Hausdorff ; lemme de recouvrement de Vitali ; espaces de Lorentz ; domaines épais uniformes ; domaines plats de Reifenberg ; estimations de décroissance ; estimations de Lorentz-Morrey ; estimations capacitaires ; équations de milieu poreux ; équations de type Riccati.

NONLINEAR POTENTIAL THEORY AND QUASILINEAR EQUATIONS WITH MEASURE DATA

Summary

This thesis is concerned to the existence and regularity of solutions to nonlinear elliptic, parabolic and Hessian equations with measure, and criteria for the existence of large solutions to some nonlinear elliptic and parabolic equations.

List of preprints and publications in this thesis

1. With M. F. Bidaut-Véron, L. Véron ; *Quasilinear Lane-Emden equations with absorption and measure data,* Journal des Mathématiques Pures et Appliquées, **102**, 315-337 (2014).

2 With L. Véron ; *Quasilinear and Hessian type equations with exponential reaction and measure data,* Archive for Rational Mechanics and Analysis, **214**, 235-267 (2014).

3 With L. Véron ; *Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption,* 17 pages, Submitted, arXiv :1308.2956.

4 With M. F. Bidaut-Véron ; *Stability properties for quasilinear parabolic equations with measure data,* 29 pages, to appear in Journal of European Mathematical Society, arXiv :1409.1518.

5 Avec M. F. Bidaut-Véron ; *Evolution equations of p-Laplace type with absorption or source terms and measure data*, 21 pages, to appear in Communications in Contemporary Mathematics, arXiv :1409.1520.

6 *Potential estimates and quasilinear parabolic equations with measure data,* 118 pages, arXiv :1405.2587v1.

7 With L. Véron ; *Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain,* 29 pages, submitted, arXiv :1406.3850.

8 With M. F. Bidaut-Véron ; Pointwise estimates and existence of solutions of porous medium and p-Laplace evolution equations with absorption and measure data, 27 pages, submitted, arXiv :1407.2218.

Key words : quasilinear elliptic equations, quasilinear parabolic equations ; Renormalized solutions ; maximal solutions ; large solutions ; Wolff potential ; Riesz potential ; Bessel Potential ; Maximal potential ; Heat kernel ; parabolic Bessel kernel ; Radon measures ; Lorentz-Bessel capacities ; Bessel capacities ; Hausdorff capacities ; Vitial covering Lemma ; Lorentz spaces ; uniformly thick domain ; Reifenberg flat domain ; decay estimates ; Lorentz-Morrey estimates ; capacitary estimates ; Porous medium equations ; Riccati type Equations.

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Introduction générale

This thesis is devoted to study the following types of problems :
- Quasilinear elliptic and Hessian equations with measure data,
- Quasilinear parabolic equations with measure data,
- Wiener type criteria for existence of large solutions to nonlinear elliptic and parabolic equations with absorption.

0.1 Quasilinear elliptic and Hessian equations with measure data

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain containing 0 and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We assume that for a.e $x \in \Omega$, $r \mapsto g(x, r)$ is nondecreasing and odd. In Chapter 1, we consider the following problem

$$\begin{aligned} -\Delta_p u + g(x, u) &= \omega \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{0.1.1}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $(1 < p < N)$, is the p-Laplacian and ω is a bounded Radon measure in Ω. When $p = 2$ and $g(x, u) = |u|^{q-1}u$ the problem has been considered by Baras and Pierre [3]. They proved that the corresponding problem to (0.1.1) admits a solution if and only if the measure ω is absolutely continuous with respect to (w.r.t) the Bessel capacity $\text{Cap}_{2,q'}$, $q' = q/(q-1)$. Here, $\text{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$, i.e,

$$\text{Cap}_{2,q'}(E) = \inf\{||\varphi||^{q'}_{W^{2,q'}(\mathbb{R}^N)} : \varphi \in S(\mathbb{R}^N), \varphi \geq 1 \text{ in a neighborhood of } E\},$$

for any compact $E \subset \mathbb{R}^N$.

We utilize Kilpelainen and Malý's result [12] (also see [11, 18]) to derive a pointwise estimate of solutions to equation $-\Delta_p u = \omega$ involving the Wolff potential $\mathbf{W}^r_{1,p}[|\omega|]$ and nonlinear potential theory for investigating problem (0.1.1), where the Wolff potential is defined by

$$\mathbf{W}^r_{1,p}[|\omega|](x) = \int_0^r \left(\frac{|\omega|(B_\rho(x))}{\rho^{N-p}}\right)^{1/(p-1)} \frac{d\rho}{\rho} \quad \text{for all } x \in \mathbb{R}^N.$$

We introduce a new suitable class of Bessel capacities associated problem (0.1.1). If $\mathbf{G}_\alpha$ is the Bessel kernel of order $\alpha > 0$ and $L^{s,q}(\mathbb{R}^N)$ is the Lorentz space with order (s, q), then capacity $\mathrm{Cap}_{\mathbf{G}_\alpha,s,q}$ of set Borel set $E\subset\mathbb{R}^N$ is defined by

$$\mathrm{Cap}_{\mathbf{G}_\alpha,s,q}(E)=\inf\{||f||^s_{L^{s,q}(\mathbb{R}^N)}:\ f\geq 0,\ \mathbf{G}_\alpha * f\geq 1\ \text{ on } E\}$$

for any Borel set E. When $q=s$, we denote $\mathrm{Cap}_{\mathbf{G}_2,q,s}$ by $\mathrm{Cap}_{\mathbf{G}_2,q}$. It is well known that the capacity $\mathrm{Cap}_{\mathbf{G}_2,q'}$ is equivalent to $\mathrm{Cap}_{2,q'}$.

In Chapter 1, we show that the problem (0.1.1) has a solution if one of the following cases is satisfied :

a) $g(x,s)=|x|^{-\beta}|s|^{q-1}s$ and ω is absolutely continuous w.r.t $\mathrm{Cap}_{\mathbf{G}_p,\frac{Nq}{Nq-(p-1)(N-\beta)},\frac{q}{q+1-p}}$,

b) $g(x,s)=|x|^{-\beta}G(s)$, G satisfies $\int_1^\infty G(s)s^{-q-1}ds<\infty$ and ω is absolutely continuous w.r.t $\mathrm{Cap}_{\mathbf{G}_p,\frac{Nq}{Nq-(p-1)(N-\beta)},1}$,

c) $g(x,s)=\mathrm{sign}(s)(e^{\tau|s|^\lambda}-1)$ and $|\omega|\leq f+\nu$ where $f\in L^1_+(\Omega)$, ν is a nonnegative bounded Radon measure which $||\mathbf{M}^{\frac{(p-1)(\lambda-1)}{\lambda}}_{p,2\mathrm{diam}(\Omega)}[\nu]||_{L^\infty(\Omega)}$ is small enough.

Here a solution of (0.1.1) is understood in the sense of renormalized (see Definition 1.3.1 in Chapter 1) and we always assume that $0\leq\beta<N, q>p-1,\tau>0,\lambda\geq 1$ and $\mathbf{M}^\eta_{\alpha,r}[\nu]$, $\eta>0, 0<\alpha<N, r>0$ is defined by

$$\mathbf{M}^\eta_{\alpha,r}[\nu](x)=\sup_{0<\rho<r}\frac{\nu(B_\rho(x))}{\rho^{N-\alpha}h_\eta(\rho)},$$

for all $x\in\mathbb{R}^N$ with $h_\eta(\rho)=\min\{(-\ln\rho)^{-\eta},(\ln 2)^{-\eta}\}$. When $p=2,\beta=0$, we obtain Baras and Pierre's sufficient condition in case **a)**.

In Chapter 2, we are concern with the following problem

$$\begin{aligned}-\Delta_p u&=g(u)+\omega \quad \text{in } \Omega,\\ u&=0 \qquad\qquad \text{in } \partial\Omega,\end{aligned}\tag{0.1.2}$$

where ω is a nonnegative bounded Radon measure in Ω and $g(u)\sim e^{a|u|^\beta}$, $a>0,\beta\geq 1$.

The case where g is a power function, i.e $g(u)=u^q$ for $q>p-1$ has been studied by Phuc and Verbitsky in [18]. They established a sufficient and necessary conditions for the existence of solutions of problem (0.1.2) expressed in terms of the capacity $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$. For example, if ω has compact support in Ω, then a sufficient and necessary condition has the following form

$$\omega(E)\leq C\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E)\qquad \text{for all compact set } E\subset\Omega$$

where C is a constant only defending on N,p,q and $d(\mathrm{supp}(\omega),\partial\Omega)$. Their construction is based upon sharp estimates from above and below of solutions of the problem $-\Delta_p u=\omega$ combined with a deep analysis of the Wolff potential.

We give a new approach in order to treat analogous questions for problem (0.1.2) in the case exponential function. We obtain a sufficient condition expressed in terms of the fractional maximal potential $\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{p,2\mathrm{diam}(\Omega)}[\omega]$ and a necessary condition expressed in terms of Orlicz capacities, see Theorem 2.1.1 in Chapter 2. We also establish this results in the case $\Omega=\mathbb{R}^N$.

Besides, in [22, 23, 24], Trudinger and Wang developed the theory of the $k-$Hessian measure and Labutin [13] obtained sharp estimates of solution of $k-$Hessian equation expressed in terms of the Wolff potential. Solutions of $k-$Hessian equation inherit almost all of properties from solutions to $p-$laplace equation. For this reason, we obtained analogous results for (0.1.2) when $p-$laplacian operator is replaced by the $k-$Hessian operator, see Theorem 2.1.3 and Theorem 2.1.4 in Chapter 2.

Furthermore, we also establish existence results for a general Wolff potential equation under the form

$$u = \mathbf{W}^R_{\alpha,p}[g(u)] + f \text{ in } \mathbb{R}^N,$$

where $0 < R \leq \infty$, $0 < \alpha p < N$ and f is a positive integrable function.

0.2 Quasilinear parabolic equations with measure data

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$ and $\Omega_T = \Omega \times (0,T)$, $T > 0$. We study the problem

$$\begin{aligned} \partial_t u - \operatorname{div}(\mathbf{A}_p(x,t,\nabla u)) &= \mu \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= \sigma \quad \text{in } \Omega, \end{aligned} \tag{0.2.1}$$

where μ is a bounded Radon measure in Ω_T, σ is an integrable function in Ω and A_p is a Carathéodory function on $\Omega_T \times \mathbb{R}^N$, such that $u \mapsto -\operatorname{div}(\mathbf{A}_p(x,t,\nabla u))$ is a nonlinear monotone and coercive mapping from the space $L^p(0,T;W^p_0(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ for $p > 1$.

It is well known that for any bounded Radon measure μ in Ω_T can be written under the form

$$\mu = f - \operatorname{div} g + h_t + \mu_s,$$

where $f \in L^1(\Omega_T)$, $g \in (L^{p'}(\Omega_T))^N$, $h \in L^p(0,T,W^{1,p}_0(\Omega))$ and μ_s is a bounded Radon measure in Ω_T with support on a set of zero $p-$parabolic capacity, proved in [7]. In [17], Petitta gave the definition of a renormalized solution for problem (0.2.1) associated above decomposition and proved that a renormalized solution exists for $p > \frac{2N+1}{N+1}$. This condition ensures that the gradient of a renormalized solution belongs to $L^1(\Omega_T)$.

In Chapter 3 (Theorem 3.2.1), we prove a stability Theorem for renormalized solutions of problem (0.2.1) with $p > \frac{2N+1}{N+1}$, extending the results of Dal Maso, Murat, Orsina and Prignet [5] for the elliptic case. More precisely, if u_n is a renormalized of problem (0.2.1) where $\sigma = \sigma_n \in L^1(\Omega)$ and

$$\mu = \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{s,n},$$

with $f_n \in L^1(\Omega_T), g_n \in (L^{p'}(\Omega_T))^N, h_n \in L^p(0,T,W^{1,p}_0(\Omega))$ and $\mu_{s,n}$ is a bounded Radon measure in Ω_T with support on a set of zero $p-$parabolic capacity and if σ_n converges to σ in $L^1(\Omega)$ and measure $\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{s,n}$ converges to $\mu = f - \operatorname{div} g + h_t + \mu_s$ in for some sense then u_n converges a.e in Ω_T to a renormalized solution u of problem (0.2.1) with data μ, σ. Moreover, $T_k(u_n - h_n)$ converges $T_k(u - h)$ in $L^p(0,T,W^{1,p}_0(\Omega))$ for any $k > 0$.

We apply this theorem and use the results mentioned in section 1 in order to solve the following equations

$$\begin{array}{rl} \partial_t u - \Delta_p u \pm g(x,u) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma & \text{in } \Omega, \end{array} \tag{0.2.2}$$

where Radon measure μ has a good behavior in time i.e $|\mu| \leq \omega \otimes f$ with nonnegative bounded Radon measure ω in Ω, $f \in L^1_+((0,T))$ and $\sigma \in L^1(\Omega)$ and g is as in section 1.

In [8], Duzaar and Mingione gave a local pointwise estimate from above of solutions to equation $\partial_t u - \operatorname{div}(\mathbf{A}_2(x,t,\nabla u)) = \mu$ involving the Riesz parabolic potential

$$\mathbb{I}_2^r[|\mu|](x,t) = \int_0^r \frac{|\mu|(B_\rho(x) \times (t-\rho^2, t+\rho^2))}{\rho^N} \frac{d\rho}{\rho},$$

for all $(x,t) \in \mathbb{R}^{N+1}$, where $\mathbf{A}_2$ is $\mathbf{A}_p$ with $p=2$ and satisfies some natural conditions. On the other hand, we always have $||\mathbb{I}_2^r[|\mu|]||_{L^s(\mathbb{R}^{N+1})} \asymp ||\mathcal{G}_2 * |\mu|||_{L^s(\mathbb{R}^{N+1})}$ where $s > 1, r > 0$ and $\mathcal{G}_2$ is the parabolic Bessel kernel of order 2, i.e.

$$\mathcal{G}_2(x,t) = \frac{\chi_{(0,\infty)}(t)}{(4\pi t)^{N/2}} \exp\left(-t - \frac{|x|^2}{4t}\right) \quad \text{for all } \ (x,t) \in \mathbb{R}^{N+1}.$$

These are our motivation in Chapter 4 for developing nonlinear parabolic potential theory.

We use this theory to solve the following equations

$$\begin{array}{rl} \partial_t u - \operatorname{div}(\mathbf{A}_2(x,t,\nabla u)) \pm |u|^{q-1}u = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma & \text{in } \Omega, \end{array} \tag{0.2.3}$$

where μ, σ are bounded Radon measures and $q \in (1,\infty)$. More precisely, problem (0.2.3) with absorption (i.e in case sign " + ") has a solution if μ, σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{\mathcal{G}_2,q'}$, $\mathrm{Cap}_{\mathbf{G}_{2/q},q'}$ respectively, see Theorem 4.2.8 in Chapter 4. Where the capacity $\mathrm{Cap}_{\mathcal{G}_2,q'}$ of a Borel set $E \subset \mathbb{R}^{N+1}$ is defined by

$$\mathrm{Cap}_{\mathcal{G}_2,p}(E) = \inf\left\{ \int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), \mathcal{G}_2 * f \geq \chi_E \right\}.$$

Problem (0.2.3) with source (i.e in case sign " − ") has a solution if

$$|\mu|(E) \leq C\mathrm{Cap}_{\mathcal{G}_2,q'}(E) \ \text{ and } \ |\sigma|(O) \leq C\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, for some a constant C.

When $A_2(x,t,\nabla u) = \nabla u$, two previous results become Baras and Pierre's results in [2, 4].

In Chapter 4, we also study the global gradient estimates for quasilinear parabolic equation (0.2.1) in case $p=2$. We obtain minimal conditions on the boundary of Ω and on the nonlinearity $\mathbf{A}_2$ so that the following statement holds

$$|||\nabla u|||_{\mathcal{K}} \leq C||\mathbb{M}_1[\nu]||_{\mathcal{K}} \ \text{ with } \nu = |\mu| + |\sigma| \otimes \delta_{\{t=0\}},$$

here the constant C does not depend on u and μ, σ and $\mathbb{M}_1[\nu]$ is the first order fractional maximal parabolic potential of ν defined by

$$\mathbb{M}_1[\nu](x,t) = \sup_{\rho>0} \frac{\nu(B_\rho(x) \times (t-\rho^2, t+\rho^2))}{\rho^{N+1}} \quad \text{for all} \quad (x,t) \in \mathbb{R}^{N+1},$$

and $\mathcal{K}$ is a function space. The same question is as above for the elliptic framework studied by N. C. Phuc in [19, 20, 21].

First, we take $\mathcal{K} = L^{p,s}(\Omega_T)$ for $1 \leq p \leq 2$ and $0 < s \leq \infty$ under a capacity density condition on the domain Ω where $L^{p,s}(\Omega_T)$ is the Lorentz space. The capacity density condition is that the complement of Ω satisfies uniformly $2-$thick. We remark that under this condition, the Sobolev embedding $H_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ for $N > 2$ is valid and it is fulfilled by any domain with Lipschitz boundary, or even of corkscrew type.

Next, in order to obtain shaper results, we take $\mathcal{K} = L^{q,s}(\Omega_T, dw)$, the weighted Lorentz spaces with weight in the Muckenhoupht class A_∞ for $q \geq 1$, $0 < s \leq \infty$, we require some stricter conditions on the domain Ω and nonlinearity $\mathbf{A}_2$. A condition on Ω is flat enough in the sense of Reifenberg, essentially, that at boundary point and every scale the boundary of domain is between two hyperplanes at both sides (inside and outside) of the domain by a distance which depends on the scale. Conditions on $\mathbf{A}_2$ are that the BMO type of $\mathbf{A}_2$ with respect to the $x-$variable is small enough and the derivative of $\mathbf{A}_2(x,t,\zeta)$ with respect to ζ is uniformly bounded. By choosing an appropriate weight we obtained some new estimates, in particular, Lorentz-Morrey estimates involving "calorie" and global capacitary estimates.

Finally, thanks to these estimates, we prove the existence of solutions of the quasilinear Riccati type parabolic equation :

$$\begin{aligned} \partial_t u - \operatorname{div}(\mathbf{A}_2(x,t,\nabla u)) &= |\nabla u|^q + \mu \quad \text{in} \quad \Omega_T, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) &= \sigma \quad \text{in } \Omega. \end{aligned} \tag{0.2.4}$$

For example, problem (0.2.4) has a solution if there exists $\varepsilon > 0$ such that

$$(|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(E) \leq C\mathrm{Cap}_{\mathcal{G}_1,(q+\varepsilon)'}(E)$$

holds for any compact $E \subset \mathbb{R}^{N+1}$ where C is a constant small enough, where $\mathcal{G}_1$ is the parabolic Bessel kernel of first order, i.e,

$$\mathcal{G}_1(x,t) = C_1 \frac{\chi_{(0,\infty)}(t)}{t^{(N+1)/2}} \exp\left(-t - \frac{|x|^2}{4t}\right) \quad \text{for all} \quad (x,t) \quad \text{in} \quad \mathbb{R}^{N+1},$$

with $C_1 = \left((4\pi)^{N/2}\Gamma(1/2)\right)^{-1}$ and the capacity $\mathrm{Cap}_{\mathcal{G}_1,(q+\varepsilon)'}$ is defined as the capacity $\mathrm{Cap}_{\mathcal{G}_2,q'}$.

In Chapter 5, we solve problem (0.2.2) with absorption term in the case $p > 2$ without all restriction on data μ by using a result in [15] of a pointwise estimate for solutions to problem (0.2.2) with $g \equiv 0$ and theory of parabolic potential introduced in Chapter 4. Besides, we also prove that the porous medium equation with absorption term

$$\begin{aligned} \partial_t u - \Delta(|u|^{m-1}u) + |u|^{q-1}u &= \mu \quad \text{in} \quad \Omega_T, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) &= \sigma \quad \text{in } \Omega, \end{aligned}$$

admits a distribution solution for $q > \max\{m, 1\}$ and $m > \frac{N-2}{N}$ if bounded Radon measures μ, σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{\mathcal{G}_2, q'}, \mathrm{Cap}_{\mathbf{G}_{2/q}, q'}$ if $m > 1$ and $\mathrm{Cap}_{\mathcal{G}_2, \frac{2q}{2(q-1)+N(1-m)}}, \mathrm{Cap}_{\mathbf{G}_{\frac{2-N(1-m)}{q}}, \frac{2q}{2(q-1)+N(1-m)}}$ if $\frac{N-2}{N} < m \le 1$, respectively.

0.3 Wiener criteria for existence of large solutions to elliptic and parabolic equations with absorption

In Chapter 6, we study the existence of solutions to the following problems

$$\begin{array}{l} -\Delta_p u + u^q = 0 \text{ in } \Omega, \\ \lim_{\delta \to 0} \inf_{B_\delta(x)} u = \infty \text{ for all } x \in \partial\Omega, \end{array} \tag{0.3.1}$$

and

$$\begin{array}{l} -\Delta_p u + e^u - 1 = 0 \text{ in } \Omega, \\ \lim_{\delta \to 0} \inf_{B_\delta(x)} u = \infty \text{ for all } x \in \partial\Omega, \end{array} \tag{0.3.2}$$

where $N \ge 2$, $1 < p < N$, $q > p - 1$ and Ω is a bounded open set in $\mathbb{R}^N$. Solutions to problems (0.3.1) and (0.3.2) are called *large solutions*.

It is well known that problems (0.3.1) and (0.3.2) have unique solutions for any bounded smooth domain Ω. Moreover, it is classical that problem (0.3.1) has a solution in the case $q < \frac{N(p-1)}{N-p}$ for any bounded open set Ω. When $N \ge 3$ and $p = 2, q \ge \frac{N}{N-2}$, a necessary and sufficient condition for the existence of large solution of (0.3.1) expressed in term of Wiener test, is

$$\int_0^1 \frac{\mathrm{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega. \tag{0.3.3}$$

In the case $q = 2$ it was obtained by probabilistic methods based upon the Brownian sake by Dhersin and Le Gall [6], this method could be extended for $\frac{N}{N-2} \le q \le 2$ by using ideas from [9, 10]. In the general case it was proved by Labutin by purely analytic methods [14].

Our main purpose of Chapter 6 is to establish a sufficient condition for the existence of solutions to problems (0.3.1) and (0.3.2) for any $q > p - 1$ and $N \ge 2$. More precisely, a sufficient condition associated (0.3.1) is

$$\int_0^1 \left(\frac{\mathrm{Cap}_{\mathbf{G}_p, \frac{q_1}{q_1 - p + 1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \tag{0.3.4}$$

for some $q_1 > \frac{Nq}{p}$ and associated (0.3.2) is

$$\int_0^1 \left(\frac{\mathcal{H}^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \ \ \forall x \in \partial\Omega, \tag{0.3.5}$$

where $\mathcal{H}^{N-p}$ is the $(N-p)-$ dimensional Hausdorff capacity in a bounded set of $\mathbb{R}^N$. We can see that condition (0.3.5) implies (0.3.4). In view of (0.3.3), then the condition (0.3.4) is not optimal in the case $p = 2$. Furthermore, we also establish behavior of high order gradient of the solution to equation (0.3.1) near boundary of Ω, where Ω is a bounded smooth domain.

In Chapter 7, we study analogous questions associated parabolic equation :

$$\begin{aligned} &\partial_t u - \Delta u + u^q = 0 \;\text{ in } O, \\ &\lim_{\delta\to 0} \inf_{O\cap Q_\delta(x,t)} u = \infty \;\text{ for all } x \in \partial_p O, \end{aligned} \tag{0.3.6}$$

where $N \geq 2$, $q \geq \frac{N+2}{N}$, O is a non-cylindrical bounded open set $O \subset \mathbb{R}^N$ and $\partial_p O$ is the parabolic boundary of O, i.e, the set all of points $X = (x,t) \in \partial O$ such that the intersection of the cylinder $Q_\delta(x,t) := B_\delta(x) \times (t-\delta^2, t)$ with O^c is not empty for any $\delta > 0$. When O is a cylindrical i.e $O = \Omega \times (a,b)$ for some bounded open set Ω in $\mathbb{R}^N$, Véron [25] showed that if the problem (0.3.1) in case $p = 2$ has a solution, then (0.3.6) does too.

We extend Labutin's idea in [14] to treat problem (0.3.6). Namely, we obtain a necessary and a sufficient condition for the existence of solutions to problem (0.3.6) in a bounded non-cylindrical domain $O \subset \mathbb{R}^{N+1}$, as follows : the necessary condition is

$$\int_0^1 \frac{\mathrm{Cap}_{\mathcal{G}_2,q'}(O^c \cap Q_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho} = \infty \;\forall\; (x,t) \in \partial_p O, \tag{0.3.7}$$

the sufficient condition is

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{\mathcal{G}_2,q'}\left(O^c \cap \left(B_{r_k}(x) \times \left(t - 1168 r_k^2, t - 1136 r_k^2\right)\right)\right)}{r_k^N} = \infty, \;\forall\; (x,t) \in \partial_p O \tag{0.3.8}$$

where $r_k = 4^{-k}$, and $N \geq 3$ when $q = \frac{N+2}{N}$.

We also obtain a sufficient condition for the existence of solutions to equation (0.3.6) in a bounded set of $\mathbb{R}^{N+1}$ when replaced u^q by $e^u - 1$, which is (0.3.8) where $\mathrm{Cap}_{\mathcal{G}_2,q'}$ is replaced by $\mathcal{PH}^N$ the parabolic $N-$dimensional Hausdorff capacity.

Finally, we apply our results of problems (0.3.1) and (0.3.6) to some viscous Hamilton-Jacobi equations : $-\Delta_p u + a_1|\nabla u|^{q_1} + b_1 u^{p-1} = 0$ for $a_1, b_1 > 0$, $p - 1 < q_2 < p \leq 2$ and $\partial_t u - \Delta u + a_2|\nabla u|^{q_2} + b_2 u^{q_3} = 0$ for $a_2, b_2 > 0$, $1 < q_2 < 2$ and $q_3 > 1$.

Bibliographie

[1] R.J. Bagby ; *Lebesgue spaces of parabolic potentials*, Ill. J. Math. **15**, 610-634 (1971).

[2] P. Baras and M. Pierre ; *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[3] P. Baras, M. Pierre ; *Singularités éliminables pour des équations semi linéaires*, Ann. Inst. Fourier Grenoble **34**, 185-206 (1984).

[4] P. Baras, M. Pierre ; *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, 185-212 (1985).

[5] G. Dal Maso, F. Murat, L. Orsina, A. Prignet ; *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa, **28**, 741-808 (1999).

[6] J. S. Dhersin and J. F. Le Gall ; *Wiener's test for super-Brownian motion and the Brownian snake*, Probab. Theory Relat. Fields **108**, 103-29 (1997).

[7] Droniou J., Porretta A. and Prignet A. ; *Parabolic capacity and soft measures for nonlinear equations,* Potential Anal. **19**, 99-161 (2003).

[8] F. Duzaar, G. Mingion ;, *Gradient estimate via non-linear potentials*, the American Journal of Mathematics, **133**, no. 4, 1093-1149 (2011).

[9] E. B. Dynkin, S. E. Kuznetsov ; *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations.* Comm. Pure Appl. Math., **51**, 897-936 (1998).

[10] E. B. Dynkin, S. E. Kuznetsov ; *Superdiffusions and removable singularities for quasilinear partial differential equations.* Comm. Pure Appl. Math., **49**, 125-176 (1996).

[11] T. Kilpelainen, J. Malý ; *Degenerate elliptic equation with measure data and nonlinear potentials*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **19**, 591-613 (1992).

[12] T. Kilpelainen, J. Malý ; *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).

[13] D. Labutin ; *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111**, 1-49, (2002).

[14] D. Labutin ; *Wiener regularity for large solutions of nonlinear equations*, Ark. Mat. **41**, no. 2, 307-39 (2003).

[15] Liskevich, Vitali ; Skrypnik, Igor I. ; Sobol, Zeev ; *Estimates of solutions for the parabolic p-Laplacian equation with measure via parabolic nonlinear potentials.* Commun. Pure Appl. Anal. **12**, no. 4, 1731-1744 (2013).

[16] Liskevich, Vitali ; Skrypnik, Igor I. ; *Pointwuse estimates for solutions to the porous medium equation with measure as a forcing term.* Isreal journal of mathematics **194** , 259-275 (2013).

[17] F. Petitta ; *Renormalized solutions of nonlinear parabolic equations with general measure data,* Ann. Math. Pura Appl.**187**, 563-604 (2008).

[18] N. C. Phuc, I. E. Verbitsky ; *Quasilinear and Hessian equations of Lane-Emden type*, Annals of Mathematics. **168**, 859-914 (2008).

[19] N.C.Phuc ; *Global integral gradient bounds for quasilinear equations below or near the natural exponent.* To appear in Arkiv for Matematik.

[20] N. C. Phuc ; *Nonlinear Muckenhoupt-Wheeden type bounds on Reifenberg flat domains*, with applications to quasilinear Riccati type equations. To appear in Advances in Mathematics.

[21] N. C. Phuc ; *Morrey global bounds and quasilinear Riccati type equations below the natural exponent.* To appear in Journal des Mathématiques Pures et Appliquées.

[22] N. S. Trudinger and X.J. Wang ; *Hessian measures*, Topological Methods in Nonlinear Analysis, **10**, 225-239 (1997).

[23] N. S. Trudinger, X.J. Wang ; *Hessian measures II*, Annals of Mathematics,**150**, 579-604 (1999).

[24] N. S. Trudinger, X. J. Wang ; *Hessian measures III*, Journal of Functional Analysis **193**, 1-23 (2002).

[25] L. Véron ; *A note on maximal solutions of nonlinear parabolic equations with absorption*, Asymptotic Analysis **72**, 189-200 (2011).

Chapitre 1

Quasilinear Lane-Emden equations with absorption and measure data

Abstract [1]

We study the existence of solutions to the equation $-\Delta_p u + g(x,u) = \mu$ when $g(x,.)$ is a nondecreasing function and μ a measure. We characterize the good measures, i.e. the ones for which the problem has a renormalized solution. We study particularly the cases where $g(x,u) = |x|^{-\beta}|u|^{q-1}u$ and $g(x,u) = \mathrm{sign}(u)(e^{\tau|u|^{\lambda}} - 1)$. The results state that a measure is good if it is absolutely continuous with respect to an appropriate Lorentz-Bessel capacities.

1. Journal des Mathématiques Pures et Appliquées, **102**, 315-337 (2014).

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing 0 and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We assume that for almost all $x \in \Omega$, $r \mapsto g(x,r)$ is nondecreasing and odd. In this article we consider the following problem

$$\begin{array}{rl} -\Delta_p u + g(x,u) = \mu & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{array} \tag{1.1.1}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $(1 < p < N)$, is the p-Laplacian and μ a bounded measure. A measure for which the problem admits a solution, in an appropriate class, is called a *good measure*. When $p = 2$ and $g(x,u) = g(u)$ the problem has been considered by Benilan and Brezis [3] in the subcritical case that is when any bounded measure is good. They prove that such is the case if $N \geq 3$ and g satisfies

$$\int_1^\infty g(s) s^{-\frac{N-1}{N-2}} ds < \infty.$$

The supercritical case, always with $p = 2$, has been considered by Baras and Pierre [2] when $g(u) = |u|^{q-1}u$ and $q > 1$. They prove that the corresponding problem to (1.1.1) admits a solution (always unique in that case) if and only if the measure μ is absolutely continuous with respect to the Bessel capacity $\mathrm{Cap}_{2,q'}$ $(q' = q/(q-1))$. In the case $p \neq 2$ it is shown by Bidaut-Véron [5] that if problem (1.1.1) with $g(x,s) = |s|^{q-1}s$ $(q > p-1)$ admits a solution, then μ is absolutely continuous with respect to any capacity $\mathrm{Cap}_{p,\frac{q}{q-p+1}+\varepsilon}$ for any $\varepsilon > 0$.

In this article we introduce a new class of Bessel capacities which are modeled on Lorentz spaces $L^{s,q}$ instead of L^q spaces. If $\mathbf{G}_\alpha$ is the Bessel kernel of order $\alpha > 0$, we denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space which is the space of functions $\phi = \mathbf{G}_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $||\phi||_{\alpha,s,q} = ||f||_{s,q}$ (a norm which is defined by using rearrangements). Then we set

$$\mathrm{Cap}_{\alpha,s,q}(E) = \inf\{||f||_{s,q} : \ f \geq 0, \ \mathbf{G}_\alpha * f \geq 1 \quad \text{on } E\}$$

for any Borel set E. We say that a measure μ in Ω is absolutely continuous with respect to the capacity $\mathrm{Cap}_{\alpha,s,q}$ if ,

$$\forall E \subset \Omega, \ E \text{ Borel }, \ \mathrm{Cap}_{\alpha,s,q}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

We also introduce the Wolff potential of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ by

$$\mathbf{W}_{\alpha,s}[\mu](x) = \int_0^\infty \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t},$$

if $\alpha > 0$, $1 < s < \alpha^{-1}N$. When we are dealing with bounded domains $\Omega \subset B_R$ and $\mu \in \mathfrak{M}^+(\Omega)$, it is useful to introduce truncated Wolff potentials.

$$\mathbf{W}^R_{\alpha,s}[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}.$$

We prove the following existence results concerning

$$\begin{aligned} -\Delta_p u + |x|^{-\beta} g(u) &= \mu \quad \text{in } \ \Omega, \\ u &= 0 \quad \text{in } \ \partial\Omega, \end{aligned} \tag{1.1.2}$$

Theorem 1.1.1 *Assume $1 < p < N$, $q > p-1$ and $0 \le \beta < N$ and μ is a bounded Radon measure in Ω.*

1. *If $g(s) = |s|^{q-1}s$, then (1.1.2) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $Cap_{p,\frac{Nq}{Nq-(p-1)(N-\beta)},\frac{q}{q+1-p}}$.*

2. *If g satisfies*

$$\int_1^\infty g(s)s^{-q-1}ds < \infty, \tag{1.1.3}$$

then (1.1.2) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $Cap_{p,\frac{Nq}{Nq-(p-1)(N-\beta)},1}$.

Furthermore, in both case there holds

$$-c\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu^-](x) \le u(x) \le c\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu^+](x) \qquad \text{for almost all } x \in \Omega, \tag{1.1.4}$$

where c is a positive constant depending on p and N.

In order to deal with exponential nonlinearities we introduce for $0 < \alpha < N$ the fractional maximal operator (resp. the truncated fractional maximal operator), defined for a positive measure μ by

$$\mathbf{M}_\alpha[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}}, \quad \left(\text{resp } \mathbf{M}_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}}\right),$$

and the η-fractional maximal operator (resp. the truncated η-fractional maximal operator)

$$\mathbf{M}_\alpha^\eta[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)}, \quad \left(\text{resp } \mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)}\right),$$

where $\eta \ge 0$ and $h_\eta(t) = \min\{(-\ln t)^{-\eta}, (\ln 2)^{-\eta}\}$ for all $t > 0$.

Theorem 1.1.2 *Assume $1 < p < N$, $\tau > 0$ and $\lambda \ge 1$. Then there exists $M > 0$ depending on N, p, τ and λ such that if a measure in Ω, $\mu = \mu^+ - \mu^-$ can be decomposed as follows*

$$\mu^+ = f_1 + \nu_1 \qquad \text{and} \ \ \mu^- = f_2 + \nu_2,$$

where $f_j \in L^1_+(\Omega)$ and $\nu_j \in \mathfrak{M}^b_+(\Omega)$ $(j = 1, 2)$, and if

$$||\mathbf{M}_{p,2diam(\Omega)}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_j]||_{L^\infty(\Omega)} < M, \tag{1.1.5}$$

there exists a renormalized solution to

$$\begin{aligned} -\Delta_p u + sign(u)\left(e^{\tau|u|^\lambda} - 1\right) &= \mu \quad \text{in } \ \Omega, \\ u &= 0 \quad \text{in } \ \partial\Omega, \end{aligned} \tag{1.1.6}$$

and satisfies (1.1.4).

Our study is based upon delicate estimates on Wolff potentials and η-fractional maximal operators which are developed in the first part of this paper.

1.2 Lorentz spaces and capacities

1.2.1 Lorentz spaces

Let (X, Σ, α) be a measured space. If $f : X \to \mathbb{R}$ is a measurable function, we set $S_f(t) := \{x \in X : |f|(x) > t\}$ and $\lambda_f(t) = \alpha(S_f(t))$. The decreasing rearrangement f^* of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\}.$$

It is well known that $(\Phi(f))^* = \Phi(f^*)$ for any continuous and nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. We set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau \qquad \forall t > 0,$$

and, for $1 \le s < \infty$ and $1 < q \le \infty$,

$$||f||_{L^{s,q}} = \begin{cases} \left(\int_0^\infty t^{\frac{q}{s}}(f^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup\limits_{t>0} \operatorname{ess} t^{\frac{1}{s}} f^{**}(t) & \text{if } q = \infty. \end{cases}$$

It is known that $L^{s,q}(X, \alpha)$ is a Banach space when endowed with the norm $||.||_{L^{s,q}}$. Furthermore there holds (see e.g. [11])

$$||t^{\frac{1}{s}} f^*||_{L^q(\mathbb{R}^+, \frac{dt}{t})} \le ||f||_{L^{s,q}} \le \frac{s}{s-1}||t^{\frac{1}{s}} f^*||_{L^q(\mathbb{R}^+, \frac{dt}{t})}, \tag{1.2.1}$$

the left-hand side inequality being valid only if $s > 1$. Finally, if $f \in L^{s,q}(\mathbb{R}^N)$ (with $1 \le q, s < \infty$ and α being the Lebesgue measure) and if $\{\rho_n\} \subset C_c^\infty(\mathbb{R}^N)$ is a sequence of mollifiers, $f * \rho_n \to f$ and $(f\chi_{B_n}) * \rho_n \to f$ in $L^{s,q}(\mathbb{R}^N)$, where χ_{B_n} is the indicator function of the ball B_n centered at the origin of radius n. In particular $C_c^\infty(\mathbb{R}^N)$ is dense in $L^{s,q}(\mathbb{R}^N)$.

1.2.2 Wolff potentials, fractional and η-fractional maximal operators

If D is either a bounded domain or whole $\mathbb{R}^N$, we denote by $\mathfrak{M}(D)$ (resp $\mathfrak{M}^b(D)$) the set of Radon measure (resp. bounded Radon measures) in D. Their positive cones are $\mathfrak{M}_+(D)$ and $\mathfrak{M}_+^b(D)$ respectively. If $0 < R \le \infty$ and $\mu \in \mathfrak{M}_+(D)$ and $R \ge \operatorname{diam}(D)$, we define, for $\alpha > 0$ and $1 < s < \alpha^{-1}N$, the R-truncated Wolff-potential by

$$\mathbf{W}_{\alpha,s}^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t} \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{1.2.2}$$

If $h_\eta(t) = \min\{(-\ln t)^{-\eta}, (\ln 2)^{-\eta}\}$ and $0 < \alpha < N$, the truncated η-fractional maximal operator is

$$\mathbf{M}_{\alpha,R}^\eta[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{1.2.3}$$

If $R = \infty$, we drop it in expressions (1.2.2) and (1.2.3). In particular

$$\mu(B_t(x)) \le t^{N-\alpha}h_\eta(t)\mathbf{M}_{\alpha,R}^\eta[\mu](x). \tag{1.2.4}$$

We also define $\mathbf{G}_\alpha$ the Bessel potential of a measure μ by

$$\mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} \mathbf{G}_\alpha(x-y)d\mu(y) \qquad \forall x \in \mathbb{R}^N,$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α in $\mathbb{R}^N$.

Definition 1.2.1 *We denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space the of functions $\phi = G_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $||\phi||_{\alpha,s,q} = ||f||_{s,q}$. If we set*

$$Cap_{\alpha,s,q}(E) = \inf\{||f||_{s,q} : \ f \geq 0, \ \mathbf{G}_\alpha * f \geq 1 \quad on \ E\},$$

for any Borel set $E \subset \mathbb{R}^N$, then $Cap_{\alpha,s,q}$ is a capacity, see [1].

1.2.3 Estimates on potentials

In the sequel, we denote by $|A|$ the N-dimensional Lebesgue measure of a measurable set A and, if F, G are functions defined in $\mathbb{R}^N$, we set $\{F > a\} := \{x \in \mathbb{R}^N : F(x) > a\}$, $\{G \leq b\} := \{x \in \mathbb{R}^N : G(x) \leq b\}$ and $\{F > a, G \leq b\} := \{F > a\} \cap \{G \leq b\}$. The following result is an extension of [12, Th 1.1]

Lemma 1.2.2 *Let $0 \leq \eta < p-1$, $0 < \alpha p < N$ and $r > 0$. There exist $c_0 > 0$ depending on N, α, p, η and $\varepsilon_0 > 0$ depending on N, α, p, η, r such that, for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ with $diam(supp(\mu)) \leq r$ and $R \in (0, \infty]$, $\varepsilon \in (0, \varepsilon_0]$, $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,R)$ there holds,*

$$\begin{aligned}\Big|\Big\{\mathbf{W}^R_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p,R}[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\Big\}\Big| \\ \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2\, \varepsilon^{-\frac{p-1}{p-1-\eta}}\right) \left|\{\mathbf{W}^R_{\alpha,p}[\mu] > \lambda\}\right| \qquad (1.2.5)\end{aligned}$$

where $l(r,R) = \frac{N-\alpha p}{p-1}\left(\min\{r,R\}^{-\frac{N-\alpha p}{p-1}} - R^{-\frac{N-\alpha p}{p-1}}\right)$ if $R < \infty$, $l(r,R) = \frac{N-\alpha p}{p-1} r^{-\frac{N-\alpha p}{p-1}}$ if $R = \infty$. Furthermore, if $\eta = 0$, ε_0 is independent of r and (1.2.5) holds for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$ and $R \in (0,\infty]$, $\varepsilon \in (0, \varepsilon_0]$, $\lambda > 0$.

Proof. Case $R = \infty$. Let $\lambda > 0$; since $\mathbf{W}_{\alpha,p}[\mu]$ is lower semicontinuous, the set

$$D_\lambda := \{\mathbf{W}_{\alpha,p}[\mu] > \lambda\}$$

is open. By Whitney covering lemma, there exists a countable set of closed cubes $\{Q_i\}_i$ such that $D_\lambda = \cup_i Q_i$, $\overset{o}{Q}_i \cap \overset{o}{Q}_j = \emptyset$ for $i \neq j$ and

$$\operatorname{diam}(Q_i) \leq \operatorname{dist}(Q_i, D^c_\lambda) \leq 4 \operatorname{diam}(Q_i).$$

Let $\varepsilon > 0$ and $F_{\varepsilon,\lambda} = \Big\{\mathbf{W}_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p}[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\Big\}$. We claim that there exist $c_0 = c_0(N, \alpha, p, \eta) > 0$ and $\varepsilon_0 = \varepsilon_0(N, \alpha, p, \eta, r) > 0$ such that for any $Q \in \{Q_i\}_i$, $\varepsilon \in (0, \varepsilon_0]$ and $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r, \infty)$ there holds

$$|F_{\varepsilon,\lambda} \cap Q| \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \varepsilon^{-\frac{p-1}{p-1-\eta}} \alpha p \ln 2\right) |Q|. \qquad (1.2.6)$$

The first we show that there exists $c_1 > 0$ depending on N, α, p and η such that for any $Q \in \{Q_i\}_i$ there holds

$$F_{\varepsilon,\lambda} \cap Q \subset E_{\varepsilon,\lambda} \ \forall \varepsilon \in (0, c_1], \lambda > 0, \tag{1.2.7}$$

where

$$E_{\varepsilon,\lambda} = \left\{ x \in Q : \mathbf{W}_{\alpha,p}^{5\,\mathrm{diam}(Q)}[\mu](x) > \lambda, (\mathbf{M}_{\alpha p}^{\eta}[\mu](x))^{\frac{1}{p-1}} \leq \varepsilon\lambda \right\}. \tag{1.2.8}$$

Infact, take $Q \in \{Q_i\}_i$ such that $Q \cap F_{\varepsilon,\lambda} \neq \emptyset$ and let $x_Q \in D_\lambda^c$ such that $\mathrm{dist}(x_Q, Q) \leq 4\,\mathrm{diam}(Q)$ and $\mathbf{W}_{\alpha,p}[\mu](x_Q) \leq \lambda$. For $k \in \mathbb{N}$, $r_0 = 5\,\mathrm{diam}(Q)$ and $x \in F_{\varepsilon,\lambda} \cap Q$, we have

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = A + B,$$

where

$$A = \int_{2^k r_0}^{2^k \frac{1+2^{k+1}}{1+2^k} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \quad \text{and} \quad B = \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Since

$$\mu(B_t(x)) \leq t^{N-\alpha p} h_\eta(t) M_{\alpha p}^{\eta}[\mu](x) \leq t^{N-\alpha p} h_\eta(t) (\varepsilon\lambda)^{p-1}. \tag{1.2.9}$$

Then

$$B \leq \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{t^{N-\alpha p} h_\eta(t) (\varepsilon\lambda)^{p-1}}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \varepsilon\lambda \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t}.$$

Replacing $h_\eta(t)$ by its value we obtain $B \leq c_2 \varepsilon\lambda 2^{-k}$ after a lengthy computation where c_2 depends only on p and η. Since $\delta := (\frac{2^k}{2^k+1})^{\frac{N-\alpha p}{p-1}}$, then $1 - \delta \leq c_3 2^{-k}$ where c_3 depends only on $\frac{N-\alpha p}{p-1}$, thus

$$\begin{aligned}
(1-\delta)A &\leq c_3 2^{-k} \int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq c_3 2^{-k} \varepsilon\lambda \int_{2^k r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\
&\leq c_4 2^{-k} \varepsilon\lambda,
\end{aligned}$$

where $c_4 = c_4(N, \alpha, p, \eta) > 0$.
By a change of variables and using that for any $x \in F_{\varepsilon,\lambda} \cap Q$ and $t \in [r_0(1+2^k), r_0(1+2^{k+1})]$, $B_{\frac{2^k t}{1+2^k}}(x) \subset B_t(x_Q)$, we get

$$\delta A = \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_{\frac{2^k t}{1+2^k}}(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Therefore

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_5 2^{-k} \varepsilon\lambda + \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

with $c_5 = c_5(N,\alpha,p,\eta) > 0$. This implies

$$\int_{r_0}^{\infty}\left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le 2c_5\varepsilon\lambda + \int_{2r_0}^{\infty}\left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le (1+2c_5\varepsilon)\lambda, \qquad (1.2.10)$$

since $\mathbf{W}_{\alpha,p}[\mu](x_Q) \le \lambda$. If $\varepsilon \in (0,c_1]$ with $c_1 = (2c_5)^{-1}$ then

$$\int_{r_0}^{\infty}\left(\frac{\mu(B_t(x)}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le 2\lambda$$

which implies (1.2.7).

Now, we let $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,\infty)$. Let B_1 be a ball with radius r such that $\operatorname{supp}(\mu) \subset B_1$. We denote B_2 by the ball concentric to B_1 with radius $2r$. Since $x \notin B_2$,

$$\mathbf{W}_{\alpha,p}[\mu](x) = \int_{r}^{\infty}\left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,\infty).$$

Thus, we obtain $D_\lambda \subset B_2$. In particular, $r_0 = 5\mathrm{diam}(Q) \le 20r$.

Next we set $m_0 = \frac{\max(1,\ln(40r))}{\ln 2}$, so that $2^{-m}r_0 \le 2^{-1}$ if $m \ge m_0$. Then for any $x \in E_{\varepsilon,\lambda}$

$$\begin{aligned}\int_{2^{-m}r_0}^{r_0}\left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} &\le \varepsilon\lambda\int_{2^{-m}r_0}^{r_0}(h_\eta(t))^{\frac{1}{p-1}}\frac{dt}{t}\\ &\le \varepsilon\lambda\int_{2^{-m}r_0}^{2^{-m_0}r_0}(-\ln t)^{\frac{-\eta}{p-1}}\frac{dt}{t} + \varepsilon\lambda\int_{2^{-m_0}r_0}^{r_0}(\ln 2)^{\frac{-\eta}{p-1}}\frac{dt}{t}\\ &\le m_0\varepsilon\lambda + \frac{(p-1)((m-m_0)\ln 2)^{1-\frac{\eta}{p-1}}}{p-1-\eta}\varepsilon\lambda.\end{aligned}$$

For the last inequality we have used $a^{1-\frac{\eta}{p-1}} - b^{1-\frac{\eta}{p-1}} \le (a-b)^{1-\frac{\eta}{p-1}}$ valid for any $a \ge b \ge 0$. Therefore,

$$\int_{2^{-m}r_0}^{r_0}\left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le \frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon\lambda \quad \forall m\in\mathbb{N}, m > (\ln 2)^{-\frac{\eta}{p-1}}m_0^{\frac{p-1}{p-1-\eta}}. \qquad (1.2.11)$$

Set

$$g_i(x) = \int_{2^{-i}r_0}^{2^{-i+1}r_0}\left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t},$$

then

$$\begin{aligned}\mathbf{W}_{\alpha,p}^{r_0}[\mu](x) &\le \frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon\lambda + \mathbf{W}_{\alpha,p}^{2^{-m}r_0}[\mu](x)\\ &\le \frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon\lambda + \sum_{i=m+1}^{\infty}g_i(x),\end{aligned}$$

for all $m > m_0^{\frac{p-1}{p-1-\eta}}$. We deduce that, for $\beta > 0$,

$$\begin{aligned}|E_{\varepsilon,\lambda}| &\leq \left|\left\{x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \varepsilon\right) \lambda\right\}\right| \\ &\leq \left|\left\{x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \sum_{i=m+1}^{\infty} 2^{-\beta(i-m-1)}(1-2^{-\beta}) \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \varepsilon\right) \lambda\right\}\right| \\ &\leq \sum_{i=m+1}^{\infty} \left|\left\{x \in Q : g_i(x) > 2^{-\beta(i-m-1)}(1-2^{-\beta}) \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}} \varepsilon\right) \lambda\right\}\right|.\end{aligned} \tag{1.2.12}$$

Next we claim that

$$|\{x \in Q : g_i(x) > s\}| \leq \frac{c_6(N,\eta)}{s^{p-1}} 2^{-i\alpha p} |Q| (\varepsilon\lambda)^{p-1}. \tag{1.2.13}$$

To see that, we pick $x_0 \in E_{\varepsilon,\lambda}$ and we use the Chebyshev's inequality

$$\begin{aligned}|\{x \in Q : g_i(x) > s\}| &\leq \frac{1}{s^{p-1}} \int_Q |g_i|^{p-1} dx \\ &= \frac{1}{s^{p-1}} \int_Q \left(\int_{r_0 2^{-i}}^{r_0 2^{-i+1}} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right)^{p-1} dx \\ &\leq \frac{1}{s^{p-1}} \int_Q \frac{\mu(B_{r_0 2^{-i+1}}(x))}{(r_0 2^{-i})^{N-\alpha p}} := A.\end{aligned}$$

Thanks to Fubini's theorem, the last term A of the above inequality can be rewritten as

$$\begin{aligned}A &= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_Q \int_{\mathbb{R}^N} \chi_{B_{r_0 2^{-i+1}}(x)}(y) d\mu(y) dx \\ &= \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} \int_Q \chi_{B_{r_0 2^{-i+1}}(y)}(x) dx d\mu(y) \\ &\leq \frac{1}{s^{p-1}} \frac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} |B_{r_0 2^{-i+1}}(y)| d\mu(y) \\ &\leq c_7(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(Q + B_{r_0 2^{-i+1}}(0)) \\ &\leq c_7(N) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(B_{r_0(1+2^{-i+1})}(x_0)),\end{aligned}$$

since $Q+B_{r_0 2^{-i+1}}(0) \subset B_{r_0(1+2^{-i+1})}(x_0)$. Using the fact that $\mu(B_t(x_0)) \leq (ln\, 2)^{-\eta} t^{N-\alpha p} (\varepsilon\lambda)^{p-1}$ for all $t > 0$ and $r_0 = 5\,\mathrm{diam}(Q)$, we obtain

$$A \leq c_8(N,\eta) \frac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} (r_0(1+2^{-i+1}))^{N-\alpha p} (\varepsilon\lambda)^{p-1} \leq c_9(N,\eta) \frac{1}{s^{p-1}} 2^{-i\alpha p} |Q| (\varepsilon\lambda)^{p-1},$$

which is (1.2.13). Consequently, (1.2.12) can be rewritten as

$$\begin{aligned}|E_{\varepsilon,\lambda}| &\le \sum_{i=m+1}^{\infty} \frac{c_6(N,\eta)}{\left(2^{-\beta(i-m-1)}(1-2^{-\beta})\left(1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon\right)\lambda\right)^{p-1}} 2^{-i\alpha p}(\varepsilon\lambda)^{p-1}|Q| \\ &\le c_6(N,\eta)2^{-(m+1)\alpha p}\left(\frac{\varepsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon}\right)^{p-1}|Q|\left(1-2^{-\beta}\right)^{-p+1}\sum_{i=m+1}^{\infty}2^{(\beta(p-1)-\alpha p)(i-m-1)}.\end{aligned} \tag{1.2.14}$$

If we choose $\beta = \beta(\alpha,p)$ so that $\beta(p-1)-\alpha p < 0$, we obtain

$$|E_{\varepsilon,\lambda}| \le c_{10}2^{-m\alpha p}\left(\frac{\varepsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon}\right)^{p-1}|Q| \qquad \forall m > (\ln 2)^{-\frac{\eta}{p-1}}m_0^{\frac{p-1}{p-1-\eta}}, \tag{1.2.15}$$

where $c_{10} = c_{10}(N,\alpha,p,\eta) > 0$. Put $\varepsilon_0 = \min\left\{\frac{1}{\frac{4(p-1)}{p-1-\eta}m_0+1}, c_1\right\}$. For any $\varepsilon \in (0,\varepsilon_0]$, we choose $m \in \mathbb{N}$ such that

$$\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\varepsilon}-1\right)^{\frac{p-1}{p-1-\eta}} - 1 < m \le \left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\varepsilon}-1\right)^{\frac{p-1}{p-1-\eta}}.$$

Then

$$\left(\frac{\varepsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\varepsilon}\right)^{p-1} \le 1,$$

and

$$2^{-m\alpha p} \le 2^{\alpha p-\alpha p\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\varepsilon}-1\right)^{\frac{p-1}{p-1-\eta}}} \le 2^{\alpha p}\exp\left(-\alpha p\ln 2\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\varepsilon^{-\frac{p-1}{p-1-\eta}}\right).$$

Combining these inequalities with (1.2.15) and (1.2.7), we get (1.2.6). In the case $\eta = 0$ we still have for any $m \in \mathbb{N}$, $\lambda, \varepsilon > 0$ and $x \in E_{\varepsilon,\lambda}$

$$\mathbf{W}_{\alpha,p}^{r_0}[\mu](x) \le m\varepsilon\lambda + \sum_{i=m+1}^{\infty} g_i(x).$$

Accordingly (1.2.15) reads as

$$|E_{\varepsilon,\lambda}| \le c_{10}2^{-m\alpha p}\left(\frac{\varepsilon}{1-m\varepsilon}\right)^{p-1}|Q| \quad \forall m \in \mathbb{N}, \lambda, \varepsilon > 0 \text{ with } m\varepsilon < 1.$$

Put $\varepsilon_0 = \min\{\frac{1}{2}, c_1\}$. For any $\varepsilon \in (0,\varepsilon_0]$ and $m \in \mathbb{N}$ satisfies $\varepsilon^{-1}-2 < m \le \varepsilon^{-1}-1$, we finally get from (1.2.7)

$$|F_{\varepsilon,\lambda} \cap Q| \le |E_{\varepsilon,\lambda}| \le c_{10}2^{2\alpha p}\exp\left(-\alpha p\varepsilon^{-1}\ln 2\right)|Q| \tag{1.2.16}$$

which ends the proof in the case $R = \infty$.
Case $R < \infty$. For $\lambda > 0$, $D_\lambda = \{\mathbf{W}^R_{\alpha,p} > \lambda\}$ is open. Using again Whitney covering lemma, there exists a countable set of closed cubes $\mathcal{Q} := \{Q_i\}$ such that $\cup_i Q_i = D_\lambda$, $\overset{o}{Q}_i \cap \overset{o}{Q}_j = \emptyset$ for $i \neq j$ and $\operatorname{dist}(Q_i, D^c_\lambda) \leq 4\operatorname{diam}(Q_i)$. If $Q \in \mathcal{Q}$: is such that $\operatorname{diam}(Q) > \frac{R}{8}$, there exists a finite number n_Q of closed dyadic cubes $\{P_{j,Q}\}_{j=1}^{n_Q}$ such that $\cup_{j=1}^{n_Q} P_{j,Q} = Q$, $\overset{o}{P}_{i,Q} \cap \overset{o}{P}_{j,Q} = \emptyset$ if $i \neq j$ and $\frac{R}{16} < \operatorname{diam}(P_{j,Q}) \leq \frac{R}{8}$. We set $\mathcal{Q}' = \left\{Q \in \mathcal{Q} : \operatorname{diam}(Q) \leq \frac{R}{8}\right\}$, $\mathcal{Q}'' = \left\{P_{i,Q} : 1 \leq i \leq n_Q, Q \in \mathcal{Q}, \operatorname{diam}(Q) > \frac{R}{8}\right\}$ and $\mathcal{F} = \mathcal{Q}' \cup \mathcal{Q}''$.
For $\varepsilon > 0$ we denote again $F_{\varepsilon,\lambda} = \left\{\mathbf{W}^R_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p,R}[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\right\}$. Let $Q \in \mathcal{F}$ such that $F_{\varepsilon,\lambda} \cap Q \neq \emptyset$ and $r_0 = 5\operatorname{diam}(Q)$.
If $\operatorname{dist}(D^c_\lambda, Q) \leq 4\operatorname{diam}(Q)$, that is if there exists $x_Q \in D^c_\lambda$ such that $\operatorname{dist}(x_Q, Q) \leq 4\operatorname{diam}(Q)$ and $\mathbf{W}^R_{\alpha,p}[\mu](x_Q) \leq \lambda$, we find, by the same argument as in the case $R = \infty$, (1.2.10), that for any $x \in F_{\varepsilon,\lambda} \cap Q$ there holds

$$\int_{r_0}^{R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + c_{11}\varepsilon)\lambda, \tag{1.2.17}$$

where $c_{11} = c_{11}(N, \alpha, p, \eta) > 0$.
If $\operatorname{dist}(D^c_\lambda, Q) > 4\operatorname{diam}(Q)$, we have $\frac{R}{16} < \operatorname{diam}(Q) \leq \frac{R}{8}$ since $Q \in \mathcal{Q}''$. Then, for all $x \in F_{\varepsilon,\lambda} \cap Q$, there holds

$$\begin{aligned}\int_{r_0}^{R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq \int_{\frac{5R}{16}}^{R} \left(\frac{t^{N-\alpha p}(\ln 2)^{-\eta}(\varepsilon\lambda)^{p-1}}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= (\ln 2)^{-\frac{\eta}{p-1}} \ln \frac{16}{5}\, \varepsilon\lambda \\ &\leq 2\varepsilon\lambda. \end{aligned}\tag{1.2.18}$$

Thus, if we take $\varepsilon \in (0, c_{12}]$ with $c_{12} = \min\{1, c_{11}^{-1}\}$, we derive

$$F_{\varepsilon,\lambda} \cap Q \subset E_{\varepsilon,\lambda}, \tag{1.2.19}$$

where

$$E_{\varepsilon,\lambda} = \left\{\mathbf{W}^{r_0}_{\alpha,p}[\mu] > \lambda, \left(\mathbf{M}^\eta_{\alpha p,R}[\mu]\right)^{\frac{1}{p-1}} \leq \varepsilon\lambda\right\}.$$

Furthermore, since $x \notin B_2$,

$$\mathbf{W}^R_{\alpha,p}[\mu](x) = \int_{\min\{r,R\}}^{R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r, R).$$

Thus, if $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r, R)$ then $D_\lambda \subset B_2$ which implies $r_0 = 5\operatorname{diam}(Q) \leq 20r$.
The end of the proof is as in the case $R = \infty$. ■
In the next result we list a series of equivalent norms concerning Radon measures.

Theorem 1.2.3 *Assume $\alpha > 0$, $0 < p - 1 < q < \infty$, $0 < \alpha p < N$ and $0 < s \leq \infty$. Then there exists a constant $c_{13} = c_{13}(N, \alpha, p, q, s) > 0$ such that for any $R \in (0, \infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, there holds*

$$c_{13}^{-1} ||\mathbf{W}^R_{\alpha,p}[\mu]||_{L^{q,s}(\mathbb{R}^N)} \leq ||\mathbf{M}_{\alpha p,R}[\mu]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{13} ||\mathbf{W}^R_{\alpha,p}[\mu]||_{L^{q,s}(\mathbb{R}^N)}. \tag{1.2.20}$$

For any $R>0$, there exists $c_{14}=c_{14}(N,\alpha,p,q,s,R)>0$ such that for any $\mu\in\mathfrak{M}_+(\mathbb{R}^N)$,

$$c_{14}^{-1}||\mathbf{W}_{\alpha,p}^R[\mu]||_{L^{q,s}(\mathbb{R}^N)}\leq||\mathbf{G}_{\alpha p}[\mu]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}\leq c_{14}||\mathbf{W}_{\alpha,p}^R[\mu]||_{L^{q,s}(\mathbb{R}^N)}. \tag{1.2.21}$$

In (1.2.21), $||\mathbf{W}_{\alpha,p}^R[\mu]||_{L^{q,s}(\mathbb{R}^N)}$ *can be replaced by* $||\mathbf{M}_{\alpha p,R}[\mu]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}$.

Proof. We denote μ_n by $\chi_{B_n}\mu$ for $n\in\mathbb{N}^*$.
Step 1. We claim that

$$||\mathbf{W}_{\alpha,p}^R[\mu]||_{L^{q,s}(\mathbb{R}^N)}\leq c'_{13}||\mathbf{M}_{\alpha p,R}[\mu]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{1.2.22}$$

From Proposition 1.2.2 there exist positive constants $c_0=c_0(N,\alpha,p), a=a(\alpha,p)$ and $\varepsilon_0=\varepsilon_0(N,\alpha,p)$ such that for all $n\in\mathbb{N}^*$, $t>0$, $0<R\leq\infty$ and $0<\varepsilon\leq\varepsilon_0$, there holds

$$\left|\left\{\mathbf{W}_{\alpha,p}^R[\mu_n]>3t,(\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}}\leq\varepsilon t\right\}\right|\leq c_0\exp\left(-a\varepsilon^{-1}\right)\left|\{\mathbf{W}_{\alpha,p}^R[\mu_n]>t\}\right|. \tag{1.2.23}$$

In the case $0<s<\infty$ and $0<q<\infty$, we have

$$\left|\left\{\mathbf{W}_{\alpha,p}^R[\mu_n]>3t\right\}\right|^{\frac{s}{q}}\leq c_{15}\exp\left(-\frac{s}{q}a\varepsilon^{-1}\right)\left|\{\mathbf{W}_{\alpha,p}^R[\mu_n]>t\}\right|^{\frac{s}{q}}+c_{15}\left|\left\{(\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}}>\varepsilon t\right\}\right|^{\frac{s}{q}},$$

with $c_{15}=c_{15}(N,\alpha,p,q,s)>0$.
Multiplying by t^{s-1} and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}\int_0^\infty t^s\left|\left\{\mathbf{W}_{\alpha,p}^R[\mu_n]>3t\right\}\right|^{\frac{s}{q}}\frac{dt}{t}&\leq c_{15}\exp\left(-\frac{s}{q}a\varepsilon^{-1}\right)\int_0^\infty t^s\left|\{\mathbf{W}_{\alpha,p}^R[\mu_n]>t\}\right|^{\frac{s}{q}}\frac{dt}{t}\\&+c_{15}\int_0^\infty t^s\left|\left\{\mathbf{M}_{\alpha p,R}^\eta[\mu_n]>(\varepsilon t)^{p-1}\right\}\right|^{\frac{s}{q}}\frac{dt}{t}.\end{aligned}$$

By a change of variable, we derive

$$\begin{aligned}&\left(3^{-s}-c_{15}\exp\left(-\frac{s}{q}a\varepsilon^{-1}\right)\right)\int_0^\infty t^s\left|\{\mathbf{W}_{\alpha,p}^R[\mu_n]>t\}\right|^{\frac{s}{q}}\frac{dt}{t}\\&\qquad\leq\frac{c_{15}\varepsilon^{-s}}{p-1}\int_0^\infty t^{\frac{s}{p-1}}\left|\left\{\mathbf{M}_{\alpha p,R}^\eta[\mu_n]>t\right\}\right|^{\frac{s}{q}}\frac{dt}{t}.\end{aligned}$$

We choose ε small enough so that $3^{-s}-c_{15}\exp\left(-\frac{s}{q}a\varepsilon^{-1}\right)>0$, we derive from (1.2.1) and $\left\|t^{1/s_1}f^*\right\|_{L^{s_2}\left(\mathbb{R},\frac{dt}{t}\right)}=s_1^{1/s_2}\left\|\lambda_f^{1/s_1}t\right\|_{L^{s_2}\left(\mathbb{R},\frac{dt}{t}\right)}$ for any $f\in L^{s_1,s_2}(\mathbb{R}^N)$ with $0<s_1<\infty,0<s_2\leq\infty$

$$||\mathbf{W}_{\alpha,p}^R[\mu_n]||_{L^{q,s}(\mathbb{R}^N)}\leq c'_{13}||\mathbf{M}_{\alpha p,R}[\mu_n]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)},$$

and (1.2.22) follows by Fatou's lemma. Similarly, we can prove (1.2.22) in the case $s=\infty$.
Step 2. We claim that

$$||\mathbf{W}_{\alpha,p}^R[\mu]||_{L^{q,s}(\mathbb{R}^N)}\geq c''_{13}||\mathbf{M}_{\alpha p,R}[\mu]||^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{1.2.24}$$

For $R>0$ we have

$$\begin{aligned}\mathbf{W}^{2R}_{\alpha,p}[\mu_n](x) &= \mathbf{W}^{R}_{\alpha,p}[\mu_n](x) + \int_R^{2R}\left(\frac{\mu_n(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\ &\leq \mathbf{W}^{R}_{\alpha,p}[\mu_n](x) + \left(\frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}}\right)^{\frac{1}{p-1}}. \end{aligned} \tag{1.2.25}$$

Thus

$$\left|\left\{x:\mathbf{W}^{2R}_{\alpha,p}[\mu_n](x)>2t\right\}\right| \leq \left|\left\{x:\mathbf{W}^{R}_{\alpha,p}[\mu_n](x)>t\right\}\right| + \left|\left\{x:\frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}}>t^{p-1}\right\}\right|.$$

Consider $\{z_j\}_{i=1}^m \subset B_2$ such that $B_2\subset\bigcup_{i=1}^m B_{\frac{1}{2}}(z_i)$. Thus $B_{2R}(x)\subset\bigcup_{i=1}^m B_{\frac{R}{2}}(x+Rz_i)$ for any $x\in\mathbb{R}^N$ and $R>0$. Then

$$\begin{aligned}\left|\left\{x:\frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}}>t^{p-1}\right\}\right| &\leq \left|\left\{x:\sum_{i=1}^m\frac{\mu_n(B_{\frac{R}{2}}(x+Rz_i))}{R^{N-\alpha p}}>t^{p-1}\right\}\right| \\ &\leq \sum_{i=1}^m\left|\left\{x:\frac{\mu_n(B_{\frac{R}{2}}(x+Rz_i))}{R^{N-\alpha p}}>\frac{1}{m}t^{p-1}\right\}\right| \\ &\leq \sum_{i=1}^m\left|\left\{x-Rz_i:\frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}}>\frac{1}{m}t^{p-1}\right\}\right| \\ &= m\left|\left\{x:\frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}}>\frac{1}{m}t^{p-1}\right\}\right|.\end{aligned}$$

Moreover

$$\left(\frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}}\right)^{\frac{1}{p-1}} \leq 2\mathbf{W}^{R}_{\alpha,p}[\mu_n](x),$$

thus

$$\left|\left\{x:\frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}}>t^{p-1}\right\}\right| \leq m\left|\left\{x:\mathbf{W}^{R}_{\alpha,p}[\mu_n](x)>\frac{1}{2m^{\frac{1}{p-1}}}t\right\}\right|.$$

This leads to

$$\left|\left\{x:\mathbf{W}^{2R}_{\alpha,p}[\mu_n](x)>2t\right\}\right| \leq (m+1)\left|\left\{x:\mathbf{W}^{R}_{\alpha,p}[\mu_n](x)>\frac{1}{2m^{\frac{1}{p-1}}}t\right\}\right| \quad \forall t>0.$$

This implies

$$||\mathbf{W}^{2R}_{\alpha,p}[\mu_n]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16}||\mathbf{W}^{R}_{\alpha,p}[\mu_n]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)},$$

with $c_{16}=c_{16}(N,\alpha,p,q,s)>0$. By Fatou's lemma, we get

$$||\mathbf{W}^{2R}_{\alpha,p}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16}||\mathbf{W}^{R}_{\alpha,p}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{1.2.26}$$

On the other hand, from the identity in (1.2.25) we derive that for any $\rho\in(0,R)$,

$$\mathbf{W}^{2R}_{\alpha,p}[\mu](x) \geq \mathbf{W}^{2\rho}_{\alpha,p}[\mu](x) \geq c_{17}\sup_{0<\rho\leq R}\left(\frac{\mu(B_\rho(x))}{\rho^{N-\alpha p}}\right)^{\frac{1}{p-1}},$$

with $c_{17} = c_{17}(N, \alpha, p) > 0$, from which follows

$$\mathbf{W}^{2R}_{\alpha,p}[\mu](x) \geq c_{17} \left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{1}{p-1}}. \tag{1.2.27}$$

Combining (1.2.26) and (1.2.27) we obtain (1.2.24) and then (1.2.20). Notice that the estimates are independent of R and thus valid if $R = \infty$.
Step 3. We claim that (1.2.21) holds. By the previous result we have also

$$c_{18}^{-1}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq ||\mathbf{M}_{\alpha p,R}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{18}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{1.2.28}$$

where $c_{18} = c_{18}(N, \alpha, p, q, s) > 0$. For $R > 0$, the Bessel kernel satisfies[14, V-3-1]

$$c_{19}^{-1}\left(\frac{\chi_{B_R}(x)}{|x|^{N-\alpha p}}\right) \leq \mathbf{G}_{\alpha p}(x) \leq c_{19}\left(\frac{\chi_{B_{\frac{R}{2}}}(x)}{|x|^{N-\alpha p}}\right) + c_{19}e^{-\frac{|x|}{2}} \quad \forall x \in \mathbb{R}^N,$$

where $c_{19} = c_{19}(N, \alpha, p, R) > 0$. Therefore

$$c_{19}^{-1}\left(\frac{\chi_{B_R}}{|.|^{N-\alpha p}}\right) * \mu \leq \mathbf{G}_{\alpha p}[\mu] \leq c_{19}\left(\frac{\chi_{B_{\frac{R}{2}}}}{|.|^{N-\alpha p}}\right) * \mu + c_{19}e^{-\frac{|.|}{2}} * \mu. \tag{1.2.29}$$

By integration by parts, we get

$$\left(\frac{\chi_{B_R}}{|.|^{N-\alpha p}}\right) * \mu(x) = (N - \alpha p)\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x) + \frac{\mu(B_R(x))}{R^{N-\alpha p}} \geq (N - \alpha p)\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x),$$

which implies

$$c_{20}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq ||\mathbf{G}_{\alpha p}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}, \tag{1.2.30}$$

where $c_{20} = c_{20}(N, \alpha, p, q, s) > 0$. Furthermore $e^{-\frac{|x|}{2}} \leq c_{21}\chi_{B_{\frac{R}{2}}} * e^{-\frac{|.|}{2}}(x)$ where $c_{21} = c_{21}(N, R) > 0$, thus

$$e^{-\frac{|.|}{2}} * \mu \leq c_{21}\left(\chi_{B_{\frac{R}{2}}} * e^{-\frac{|.|}{2}}\right) * \mu = c_{21}e^{-\frac{|.|}{2}} * \left(\chi_{B_{\frac{R}{2}}} * \mu\right).$$

Since

$$\chi_{B_{\frac{R}{2}}} * \mu(x) = \mu(B_{\frac{R}{2}}(x)) \leq c_{22}\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x),$$

where $c_{22} = c_{22}(N, \alpha, p, R) > 0$, we derive with $c_{23} = c_{21}c_{22}$

$$e^{-\frac{|.|}{2}} * \mu \leq c_{23}e^{-\frac{|.|}{2}} * \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu].$$

Using Young inequality, we obtain

$$\begin{aligned} ||e^{-\frac{|.|}{2}} * \mu||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} &\leq c_{23}||e^{-\frac{|.|}{2}} * \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \\ &\leq c_{24}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}||e^{-\frac{|.|}{2}}||_{L^{1,\infty}(\mathbb{R}^N)} \\ &\leq c_{25}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}, \end{aligned} \tag{1.2.31}$$

where $c_{25} = c_{25}(N, \alpha, p, R) > 0$.
Since by integration by parts there holds as above

$$\left(\frac{\chi_{B_{\frac{R}{2}}}}{|.|^{N-\alpha p}}\right) * \mu(x) = (N - \alpha p)\mathbf{W}^{\frac{R}{2}}_{\frac{\alpha p}{2},2}[\mu](x) + 2^{N-\alpha p}\frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \leq c_{26}\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x),$$

where $c_{26} = c_{26}(N, \alpha, p) > 0$ we obtain

$$||\left(\frac{\chi_{B_R}}{||.|^{N-\alpha p}}\right) * \mu||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{27}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}}, \tag{1.2.32}$$

where $c_{27} = c_{27}(N, \alpha, p, q, s) > 0$. Thus

$$||\mathbf{G}_{\alpha p}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{28}||\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]||_{L^{\frac{q}{p-1},\frac{s}{p-1}}}, \tag{1.2.33}$$

where $c_{28} = c_{28}(N, \alpha, p, q, s, R) > 0$, follows by combining (1.2.29), (1.2.31) and (1.2.32). Then, combining (1.2.30), (1.2.33) and using (1.2.28), (1.2.20) we obtain (1.2.21). ■

Remark 1.2.4 *Proposition 5.1 in [13] is a particular case of the previous result.*

Theorem 1.2.5 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p - 1$, $0 < \alpha p < N$ and $r > 0$. Set $\delta_0 = \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2$. Then there exists $c_{29} > 0$, depending on N, α, p, η and r such that for any $R \in (0, \infty]$, $\delta \in (0, \delta_0)$, $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, any ball $B_1 \subset \mathbb{R}^N$ with radius $\leq r$ and ball B_2 concentric to B_1 with radius double B_1's radius, there holds*

$$\frac{1}{|B_2|}\int_{B_2} \exp\left(\delta\frac{\left(\mathbf{W}^{R}_{\alpha,p}[\mu_{B_1}](x)\right)^{\frac{p-1}{p-1-\eta}}}{||\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]||^{\frac{1}{p-1-\eta}}_{L^\infty(B_1)}}\right) dx \leq \frac{c_{29}}{\delta_0 - \delta} \tag{1.2.34}$$

where $\mu_{B_1} = \chi_{B_1}\mu$. Furthermore, if $\eta = 0$, c_{29} is independent of r.

Proof. Let $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ such that $M := ||\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]||_{L^\infty(B_1)} < \infty$. By Proposition 1.2.2-(1.2.5) with $\mu = \mu_{B_1}$, there exist $c_0 > 0$ depending on N, α, p, η and $\varepsilon_0 > 0$ depending on N, α, p, η and r such that, for all $R \in (0, \infty]$, $\varepsilon \in (0, \varepsilon_0]$, $t > \left(\mu_{B_1}(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r', R)$ where r' is radius of B_1 there holds,

$$\begin{aligned}&\left|\left\{\mathbf{W}^{R}_{\alpha,p}[\mu_{B_1}] > 3t, (\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}])^{\frac{1}{p-1}} \leq \varepsilon t\right\}\right| \\ &\qquad \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2\, \varepsilon^{-\frac{p-1}{p-1-\eta}}\right) \left|\{\mathbf{W}^{R}_{\alpha,p}[\mu_{B_1}] > t\}\right|.\end{aligned}$$

Since $\left(\mu_{B_1}(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r', R) \leq \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}} M^{\frac{1}{p-1}}$, thus in (1.2.5) we can choose

$$\varepsilon = t^{-1}||\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]||^{\frac{1}{p-1}}_{L^\infty(\mathbb{R}^N)} = t^{-1}M^{\frac{1}{p-1}} \quad \forall t > \max\{\varepsilon_0^{-1}, \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}}\}M^{\frac{1}{p-1}},$$

and as in the proof of Proposition 1.2.2, $\left\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > t\right\} \subset B_2$.
Then

$$\left|\left\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > 3t\right\} \cap B_2\right| \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}}\right) |B_2|. \tag{1.2.35}$$

This can be written under the form

$$|\{F > t\} \cap B_2| \leq |B_2|\chi_{(0,t_0]} + c_0 \exp\left(-\delta_0 t\right) |B_2|\chi_{(t_0,\infty)}(t), \tag{1.2.36}$$

where $F = M^{-\frac{1}{p-1-\eta}} \left(\mathbf{W}^R_{\alpha,p}[\mu_{B_1}]\right)^{\frac{p-1}{p-1-\eta}}$ and $t_0 = \left(3 \max\{\varepsilon_0^{-1}, \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}}\}\right)^{\frac{p-1}{p-1-\eta}}$.
Take $\delta \in (0, \delta_0)$, by Fubini's theorem

$$\int_{B_2} \exp\left(\delta F(x)\right) dx = \delta \int_0^\infty \exp\left(\delta t\right) |\{F > t\} \cap B_2| dt.$$

Thus,

$$\begin{aligned}\int_{B_2} \exp\left(\delta F(x)\right) dx &\leq \delta \int_0^{t_0} \exp\left(\delta t\right) dt |B_2| + c_0\delta \int_{t_0}^\infty \exp\left(-\left(\delta_0 - \delta\right) t\right) dt |B_2| \\ &\leq \left(\exp\left(\delta t_0\right) - 1\right) |B_2| + \frac{c_0\delta}{\delta_0 - \delta} |B_2|\end{aligned}$$

which is the desired inequality. ∎

Remark 1.2.6 *By the proof of Proposition 1.2.2, we see that* $\varepsilon_0 \geq \frac{c_{30}}{\max(1, \ln 40r)}$ *where* $c_{30} = c_{30}(N, \alpha, p, \eta) > 0$. *Thus,* $t_0 \leq c_{31} \left(\max(1, \ln 40r)\right)^{\frac{p-1}{p-1-\eta}}$. *Therefore,*

$$c_{29} \leq c_{32} \exp\left(c_{33} \left(\max(1, \ln 40r)\right)^{\frac{p-1}{p-1-\eta}}\right),$$

where c_{32} *and* c_{33} *depend on* N, α, p *and* η.

1.2.4 Approximation of measures

The next result is an extension of a classical result of Feyel and de la Pradelle [10]. This type of result has been intensively used in the framework of Sobolev spaces since the pioneering work of Baras and Pierre [2], but apparently it is new in the case of Bessel-Lorentz spaces. We recall that a sequence of bounded measures $\{\mu_n\}$ in Ω converges to some bounded measure μ in Ω in the *narrow topology* of $\mathfrak{M}_b(\Omega)$ if

$$\lim_{n\to\infty} \int_\Omega \phi d\mu_n = \int_\Omega \phi d\mu \qquad \forall \phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega). \tag{1.2.37}$$

Theorem 1.2.7 *Assume* Ω *is an open subset of* $\mathbb{R}^N$. *Let* $\alpha > 0$, $1 < s < \infty$, $1 \leq q < \infty$ *and* $\mu \in \mathfrak{M}_+(\Omega)$. *If* μ *is absolutely continuous with respect to* $Cap_{\alpha,s,q}$ *in* Ω, *there exists a nondecreasing sequence* $\{\mu_n\} \subset \mathfrak{M}^+_b(\Omega) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$, *with compact support in* Ω *which converges to* μ *weakly in the sense of measures. Furthermore, if* $\mu \in \mathfrak{M}^+_b(\Omega)$, *then* $\mu_n \rightharpoonup \mu$ *in the narrow topology.*

Proof. Step 1. Assume that μ has compact support. Let $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$ and $\tilde{\phi}$ its $\mathrm{Cap}_{\alpha,s,q}$-quasicontinuous representative. Since μ is abolutely continuous with respect to $\mathrm{Cap}_{\alpha,s,q}$, we can define the mapping

$$\phi \mapsto P(\phi) = \int_{\mathbb{R}^N} \tilde{\phi}^+ d\mu\lfloor_\Omega$$

where $\mu\lfloor_\Omega$ is the extension of μ by 0 in Ω^c. By Fatou's lemma, P is lower semi-continuous on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore it is convex and potitively homogeneous of degree 1. If $\mathrm{Epi}(P)$ denotes the epigraph of P, i.e.

$$\mathrm{Epi}(P) = \{(\phi,t) \in L^{\alpha,s,q}(\mathbb{R}^N) \times \mathbb{R} : t \geq P(\phi)\},$$

it is a closed convex cone. Let $\varepsilon > 0$ and $\phi_0 \in C_c^\infty$, $\phi_0 \geq 0$. Since $(\phi_0, P(\phi_0) - \varepsilon) \notin \mathrm{Epi}(P)$, there exist $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$, a and b in $\mathbb{R}$ such that

$$a + bt + \ell(\phi) \leq 0 \qquad \forall (\phi,t) \in \mathrm{Epi}(P), \tag{1.2.38}$$

$$a + b(P(\phi_0) - \varepsilon) + \ell(\phi_0) > 0. \tag{1.2.39}$$

Since $(0,0) \in \mathrm{Epi}(P)$, $a \leq 0$. Since $(s\phi, st) \in \mathrm{Epi}(P)$ for all $s > 0$, $s^{-1}a + bt + \ell(\phi) \leq 0$, which implies

$$bt + \ell(\phi) \leq 0 \qquad \forall (\phi,t) \in \mathrm{Epi}(P).$$

Finally, since $(0,1) \in \mathrm{Epi}(P)$, $b \leq 0$. But if $b = 0$ we would have $\ell(\phi) \leq -a$ for all $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$. which would lead to $\ell = 0$ and $a > 0$ from (1.2.39), a contradiction. Therefore $b < 0$. Then, we put $\theta(\phi) = -\frac{\ell(\phi)}{b}$ and derive that, for any $(\phi,t) \in \mathrm{Epi}(P)$, there holds $\theta(\phi) \leq t$, and in particular

$$\theta(\phi) \leq P(\phi) \qquad \forall \phi \in L^{\alpha,s,q}(\mathbb{R}^N). \tag{1.2.40}$$

Since $\phi \leq 0 \Longrightarrow P(\phi) = 0$, θ is a positive linear functional on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore

$$\sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ ||\phi||_{L^\infty} \leq 1}} |\theta(\phi)| = \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ ||\phi||_{L^\infty} \leq 1}} \theta(\phi) \leq \sup_{\substack{\phi \in C_c^\infty(\mathbb{R}^N) \\ ||\phi||_{L^\infty} \leq 1}} P(\phi) = P(1) = \mu(\Omega).$$

By the Riesz representation theorem, there exists $\sigma \in \mathfrak{M}_+(\mathbb{R}^N)$ such that

$$\theta(\phi) = \int_{\mathbb{R}^N} \phi d\sigma \qquad \forall \phi \in C_c^\infty(\mathbb{R}^N). \tag{1.2.41}$$

Inequality (1.2.40) implies $0 \leq \sigma \leq \mu\lfloor_\Omega$. Thus $\mathrm{supp}(\sigma) \subset \mathrm{supp}(\mu\lfloor_\Omega) = \mathrm{supp}(\mu)$ and σ vanishes on Borel subsets of $\mathrm{Cap}_{\alpha,s,q}$ capacity zero, as μ does it, besides (1.2.41) also values for all $\phi \in C^\infty(\mathbb{R}^N)$. From (1.2.39), we have

$$\int_{\mathbb{R}^N} \tilde{\phi}_0 d\sigma = \theta(\phi_0) > P(\phi_0) - \varepsilon + \frac{a}{b} \geq \int_{\mathbb{R}^N} \tilde{\phi}_0 d\mu\lfloor_\Omega - \varepsilon.$$

This implies

$$0 \leq \int_{\mathbb{R}^N} \tilde{\phi}_0 d(\mu\lfloor_\Omega - \sigma) \leq \varepsilon. \tag{1.2.42}$$

It remains to prove that $\sigma \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. For all $f \in C_c^\infty(\mathbb{R}^N)$, $f \geq 0$, there holds

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f]d\sigma = \theta(\mathbf{G}_\alpha[f]) \leq ||\theta||_{(L^{\alpha,s,q}(\mathbb{R}^N))'}||\mathbf{G}_\alpha[f]||_{L^{\alpha,s,q}(\mathbb{R}^N)}, \tag{1.2.43}$$

since $\theta = -b^{-1}\ell$ and $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. Now, given $f \in L^{s,q}(\mathbb{R}^N)$, $f \geq 0$ and a sequence of modifiers $\{\rho_n\}$, $(\chi_{B_n}f)*\rho_n \in C_c^\infty(\mathbb{R}^N)$ and $(\chi_{B_n}f)*\rho_n \to f$ in $L^{s,q}(\mathbb{R}^N)$, where χ_{B_n} is the indicator function of the ball B_n centered at the origin of radius n. Furthermore, there is a subsequence $\{n_k\}$ such that $\lim_{n_k\to\infty} \mathbf{G}_\alpha[(\chi_{B_{n_k}}f)*\rho_{n_k}](x) \to \mathbf{G}_\alpha[f](x)$, $\mathrm{Cap}_{\alpha,s,q}$-quasi everywhere. Using Fatou's lemma and lower semicontinuity of the norm

$$\begin{aligned}
\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f]d\sigma &\leq \liminf_{n_k\to\infty} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[(\chi_{B_{n_k}}f)*\rho_{n_k}]d\sigma \\
&\leq \liminf_{n_k\to\infty} ||\theta||_{(L^{\alpha,s,q}(\mathbb{R}^N))'}||\mathbf{G}_\alpha[(\chi_{B_{n_k}}f)*\rho_{n_k}]||_{L^{\alpha,s,q}(\mathbb{R}^N)} \\
&\leq ||\theta||_{(L^{\alpha,s,q}(\mathbb{R}^N))'}||\mathbf{G}_\alpha[f]||_{L^{\alpha,s,q}(\mathbb{R}^N)}.
\end{aligned}$$

Therefore (1.2.43) also holds for all $f \in L^{s,q}(\mathbb{R}^N), f \geq 0$. Consequently $\sigma \in \mathfrak{M}_b^+(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ satisfies

$$\left|\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f]d\sigma\right| \leq ||\theta||_{(L^{\alpha,s,q}(\mathbb{R}^N))'}||\mathbf{G}_\alpha[f]||_{L^{\alpha,s,q}(\mathbb{R}^N)} \qquad \forall f \in L^{s,q}(\mathbb{R}^N). \tag{1.2.44}$$

Step 2. We assume that μ has no longer compact support. Set $\Omega_n = \{x \in \Omega : \mathrm{dist}(x,\Omega^c) \geq n^{-1}, |x| \leq n\}$, then $\Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$ for $n \geq n_0$ such that $\Omega_{n_0} \neq \emptyset$. Let $\{\phi_n\} \subset C_c^\infty(\mathbb{R}^N)$ be an increasing sequence such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ in a neighborhood of $\overline{\Omega_n}$ and $\mathrm{supp}(\phi_n) \subset \Omega_{n+1}$. and let $\nu_n = \phi_n\mu$. For $n \geq n_0$ there is $\sigma_n \in \mathfrak{M}_b^+(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ with $0 \leq \sigma_n \leq \nu_n$ and

$$\frac{1}{n} > \int_\Omega \phi_n d(\nu_n - \sigma_n) \geq \int_{\Omega_n} d(\mu_n - \sigma_n) = \int_{\Omega_n} d(\mu - \sigma_n).$$

We set $\mu_n = \sup\{\sigma_1, \sigma_2, ..., \sigma_n\}$, then $\{\mu_n\}$ is nondecreasing and $\mathrm{supp}(\mu_n) \subset \Omega_{n+1}$, and $\mu_n \in \mathfrak{M}_b^+(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$. Finally, let $\phi \in C_c(\Omega)$ and $m \in \mathbb{N}^*$ such that $\mathrm{supp}(\phi) \subset \Omega_m$. For all $n \geq m$, we have

$$\left|\int_\Omega \phi d\mu_n - \int_\Omega \phi d\mu\right| \leq \left|\int_{\Omega_n} d(\mu - \mu_n)\right| ||\phi||_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{n}||\phi||_{L^\infty(\mathbb{R}^N)}.$$

Thus $\mu_n \rightharpoonup \mu$ weakly in the sense of measures.
Step 3. Assume that $\mu \in \mathfrak{M}_b^+(\Omega)$. Then $\mu_n(\Omega) \leq \mu(\Omega)$. Thus

$$\mu_n(\Omega) = \mu_n(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k).$$

Since the sequence $\{\mu_n\}$ is nondecreasing and $\lim_{k\to\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\overline{\Omega}_{k+1} \setminus \Omega_k)$ by the previous construction, we obtain by monotone convergence

$$\lim_{n\to\infty} \mu_n(\Omega) = \mu(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\Omega).$$

Next we consider $\phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega)$, then

$$\left|\int_\Omega \phi d\mu_n - \int_\Omega \phi d\mu\right| \leq \left|\int_\Omega d(\mu - \mu_n)\right| |\phi||_{L^\infty(\Omega)} \leq (\mu(\Omega) - \mu_n(\Omega))||\phi||_{L^\infty(\Omega)} \to 0.$$

Thus $\mu_n \rightharpoonup \mu$ in the narrow topology of measures. ■

As a consequence of Theorem 1.2.7 and Theorem 1.2.3 we obtain the following :

Theorem 1.2.8 *Let $p - 1 < s_1 < \infty$, $p - 1 < s_2 \leq \infty$, $0 < \alpha p < N$, $R > 0$ and $\mu \in \mathfrak{M}_+(\Omega)$. If μ is absolutely continuous with respect to the capacity $Cap_{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}$, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}^+(\Omega)$ with compact support in Ω which converges to μ in the weak sense of measures and such that $\mathbf{W}^R_{\alpha,p}[\mu_n] \in L^{s_1,s_2}(\mathbb{R}^N)$, for all n. Furthermore, if $\mu \in \mathfrak{M}^+_b(\Omega)$, μ_n converges to to μ in the narrow topology.*

Proof. By Theorem 1.2.7 there exists a nondecreasing sequence $\{\mu_n\}$ of nonnegative measures with compact support in Ω, all elements of $(L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))'$, which converges weakly to μ. If $\mu \in \mathfrak{M}^+_b(\Omega)$, the convergence holds in the narrow topology. Noting that for a positive measure σ in $\mathbb{R}^N$,

$$\mathbf{G}_{\alpha p}[\sigma] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N) \Longleftrightarrow \sigma \in (L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))',$$

it implies $\mathbf{G}_{\alpha p}[\mu_n] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N)$. Then, by Theorem 1.2.3, $\mathbf{W}^R_{\alpha,p}[\mu_n] \in L^{s_1,s_2}(\mathbb{R}^N)$. ■

1.3 Renormalized solutions

1.3.1 Classical results

Although the notion of renormalized solutions is becoming more and more present in the theory of quasilinear equations with measure data, it has not yet acquainted a popularity which could avoid us to present some of its main aspects. Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $\text{Cap}^\Omega_{1,p}$-capacity defined on a compact set $K \subset \Omega$ by

$$\text{Cap}^\Omega_{1,p}(K) = \inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \geq \chi_K, \phi \in C^\infty_c(\Omega)\right\}. \tag{1.3.1}$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $\text{Cap}^\Omega_{1,p}$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. We recall that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - \text{div}g$ where $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\}, -k\}$. We recall that if u is a measurable function defined and finite a.e. in Ω, such that $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [9].

Theorem 1.3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u = \mu \quad & in\ \Omega, \\ u = 0 \quad & on\ \partial\Omega, \end{aligned} \tag{1.3.2}$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist $\lambda_k^+, \lambda_k^- \in \mathfrak{M}_b^+(\Omega) \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\lambda_k^+ \rightharpoonup \mu_s^+$, $\lambda_k^- \rightharpoonup \mu_s^-$ in the narrow topology of measures, such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2}\nabla u \nabla\phi dx = \int_{\{|u|<k\}} \phi d\mu_0 + \int_\Omega \phi d\lambda_k^+ - \int_\Omega \phi d\lambda_k^-, \tag{1.3.3}$$

for every $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 1.3.2 *If u is a renormalized solution of problem (1.3.2) and $\mu \in \mathfrak{M}_b^+(\Omega)$, then $u \geq 0$ in Ω.*

We recall the following important results, see [9, Th 4.1, Sec 5.1].

Theorem 1.3.3 *Let $\{\mu_n\} \subset \mathfrak{M}_b(\Omega)$ be a sequence such that $\sup_n |\mu_n|(\Omega) < \infty$ and let $\{u_n\}$ be renormalized solutions of*

$$\begin{aligned} -\Delta_p u_n = \mu_n \quad & in\ \Omega, \\ u_n = 0 \quad & on\ \partial\Omega. \end{aligned} \tag{1.3.4}$$

Then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution u of $-\Delta_p u = \mu$ in the sense of distributions in Ω, for some measure $\mu \in \mathfrak{M}_b(\Omega)$, and for every $k > 0$, $k^{-1} \int_\Omega |\nabla T_k(u)|^p \leq M$ for some $M > 0$.

Finally we recall the following fundamental stability result of [9] which extends Theorem 1.3.3.

Theorem 1.3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(\Omega)$, with $\mu_0 = f - div g \in \mathfrak{M}_0(\Omega)$, $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega)$. Assume there are sequences $\{f_n\} \subset L^1(\Omega)$, $\{g_n\} \subset (L^{p'}(\Omega))^N$, $\{\eta_n^1\}, \{\eta_n^2\} \subset \mathfrak{M}_b^+(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L^1(\Omega)$, $g_n \to g$ in $L^{p'}(\Omega)$ and div g_n is bounded in $\mathfrak{M}_b(\Omega)$, $\eta_n^1 \rightharpoonup \mu_s^+$ and $\eta_n^2 \rightharpoonup \mu_s^-$ in the narrow topology. If $\mu_n = f_n - div g_n + \eta_n^1 - \eta_n^2$ and u_n is a renormalized solution of (1.3.4), then, up to a subsequence, u_n converges a.e. to a renormalized solution u of* (1.3.2). *Furthermore, $T_k(u_n) \to T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.*

1.3.2 Applications

We present below some interesting consequences of the above theorem.

Corollary 1.3.5 *Let $\mu \in \mathfrak{M}^b(\Omega)$ with compact support in Ω and $\omega \in \mathfrak{M}_b(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges weakly to $f \in L^1(\Omega)$ and $\mu_n = \rho_n * \mu$ where $\{\rho_n\}$ is a sequence of mollifiers. If u_n is a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_n + \omega \quad && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

then, up to a subsequence, u_n converges to a renormalized solution of

$$\begin{aligned} -\Delta_p u &= f + \mu + \omega \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Proof. We write $\omega = \tilde{h} - div\, \tilde{g} + \omega_s^+ - \omega_s^-$ and $\mu = h - div\, g + \mu_s^+ - \mu_s^-$, with $h, \tilde{h} \in L^1(\Omega)$, $g, \tilde{g} \in (L^{p'}(\Omega))^N$, h, g, μ_s^+ and μ_s^- with support in a compact set $K \subset \Omega$. For n_0 large enough, $\rho_n * h$, $\rho_n * g$, $\rho_n * \mu_s^+$and $\rho_n * \mu_s^-$ have also their support in a fixed compact subset of Ω for all $n \geq n_0$. Moreover $\rho_n * h \to h$ and $\rho_n * g \to g$ in $L^1(\Omega)$ and $(L^{p'}(\Omega))^N$ respectively and div$\rho_n * g \to div\, g$ in $W^{-1,p'}(\Omega)$. Therefore

$$f_n + \mu_n + \omega = f_n + \tilde{h} + \rho_n * h - div\,(\tilde{g} + \rho_n * g) + \omega_s^+ + \rho_n * \mu_s^+ - \omega_s^- - \rho_n * \mu_s^-$$

is an approximation of the measure $f + \mu + \omega$ in the sense of Theorem 1.3.4. This implies the claim. ∎

Corollary 1.3.6 *Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, and $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$ be a nondecreasing and converging to μ_i in $\mathfrak{M}_b^+(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges to some f weakly in $L^1(\Omega)$. Let $\{\vartheta_n\} \subset \mathfrak{M}^b(\Omega)$ which converges to some $\vartheta \in \mathfrak{M}_s(\Omega)$ in the narrow topology. For any $n \in \mathbb{N}$ let u_n be a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_{1,n} - \mu_{2,n} + \vartheta_n \quad && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then, up to a subsequence, u_n converges a.e. to a renormalized solution of problem

$$\begin{aligned} -\Delta_p u &= f + \mu_1 - \mu_2 + \vartheta \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The proof of this results is based upon two lemmas

Lemma 1.3.7 *For any $\mu \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ there exists $f \in L^1(\Omega)$ and $h \in W^{-1,p'}(\Omega)$ such that $\mu = f + h$ and*

$$||f||_{L^1(\Omega)} + ||h||_{W^{-1,p'}(\Omega)} + ||h||_{\mathfrak{M}^b(\Omega)} \leq 5\mu(\Omega). \tag{1.3.5}$$

Proof. Following [8] and the proof of [6, Th 2.1], one can write $\mu = \phi\gamma$ where $\gamma \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_b^+(\Omega)$ and $0 \leq \phi \in L^1(\Omega, \gamma)$. Let $\{\Omega_n\}_{n \in \mathbb{N}_*}$ be an increasing sequence of compact subsets of Ω such that $\cup_n \Omega_n = \Omega$. We define the sequence of measures $\{\nu_n\}_{n \in \mathbb{N}_*}$ by

$$\nu_1 = T_1(\chi_{\Omega_1}\phi)\gamma, \nu_n = T_n(\chi_{\Omega_n}\phi)\gamma - T_{n-1}(\chi_{\Omega_{n-1}}\phi)\gamma \quad \text{for } n \geq 2.$$

Since $\nu_k \geq 0$, then $\sum_{k=1}^{\infty} \nu_k = \mu$ with strong convergence in $\mathfrak{M}^b(\Omega)$, $||\nu_k||_{\mathfrak{M}_b(\Omega)} = \nu_k(\Omega)$ and $\sum_{k=1}^{\infty} ||\nu_k||_{\mathfrak{M}^b(\Omega)} = \mu(\Omega)$. Let $\{\rho_n\}$ be a sequence of mollifiers. We may assume that $\eta_n = \rho_n * \nu_n \in C_c^{\infty}(\Omega)$,

$$||\eta_n - \nu_n||_{W^{-1,p'}(\Omega)} \leq 2^{-n}\mu(\Omega).$$

Set $f_n = \sum_{k=1}^{n} \eta_k$, then $||f_n||_{L^1(\Omega)} \leq \sum_{k=1}^{n} ||\eta_k||_{L^1(\Omega)} \leq \sum_{k=1}^{n} ||\nu_k||_{\mathfrak{M}_b(\Omega)} \leq \mu(\Omega)$. If we define $f = \lim_{n\to\infty} f_n$, then $f \in L^1(\Omega)$ with $||f||_{L^1(\Omega)} \leq \mu(\Omega)$. Set $h_n = \sum_{k=1}^{n}(\nu_k - \eta_k)$, then $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_b(\Omega)$, $||h_n||_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$ and h_n converges strongly in $W^{-1,p'}(\Omega)$ to some h which satisfies $||h||_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$. Since $\mu = f + h$ and $||h||_{\mathfrak{M}_b(\Omega)} \leq 2\mu(\Omega)$, the result follows. ∎

Lemma 1.3.8 *Let $\mu \in \mathfrak{M}_b^+(\Omega)$. If $\{\mu_n\} \subset \mathfrak{M}_b^+(\Omega)$ is a nondecreasing sequence which converges to μ in $\mathfrak{M}_b(\Omega)$, there exist $F_n, F \in L^1(\Omega)$, $G_n, G \in W^{-1,p'}(\Omega)$ and $\mu_{n\,s}, \mu_s \in \mathfrak{M}_s(\Omega)$ such that*

$$\mu_n = \mu_{n\,0} + \mu_{n\,s} = F_n + G_n + \mu_{n\,s} \quad \text{and} \quad \mu = \mu_0 + \mu_s = F + G + \mu_s,$$

such that $F_n \to F$ in $L^1(\Omega)$, $G_n \to G$ in $W^{-1,p'}(\Omega)$ and in $\mathfrak{M}^b(\Omega)$ and $\mu_{n\,s} \to \mu_s$ in $\mathfrak{M}^b(\Omega)$, and

$$||F_n||_{L^1(\Omega)} + ||G_n||_{W^{-1,p'}(\Omega)} + ||G_n||_{\mathfrak{M}_b(\Omega)} + ||\mu_{n\,s}||_{\mathfrak{M}_b(\Omega)} \leq 6\mu(\Omega). \tag{1.3.6}$$

Proof. Since $\{\mu_n\}$ is nondecreasing $\{\mu_{n\,0}\}$ and $\{\mu_{n\,s}\}$ share this property. Clearly

$$||\mu - \mu_n||_{\mathfrak{M}^b(\Omega)} = ||\mu_0 - \mu_{n\,0}||_{\mathfrak{M}^b(\Omega)} + ||\mu_s - \mu_{n\,s}||_{\mathfrak{M}^b(\Omega)},$$

thus $\mu_{n\,0} \to \mu_0$ and $\mu_{n\,s} \to \mu_s$ in $\mathfrak{M}_b(\Omega)$. Furthermore $||\mu_{n\,s}||_{\mathfrak{M}_b(\Omega)} \leq \mu_s(\Omega) \leq \mu(\Omega)$. Set $\tilde{\mu}_{0\,0} = 0$ and $\tilde{\mu}_{n\,0} = \mu_{n\,0} - \mu_{n-1\,0}$ for $n \in \mathbb{N}_*$. From Lemma 1.3.7, for any $n \in \mathbb{N}$, one can find $f_n \in L^1(\Omega)$, $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_b(\Omega)$ such that $\tilde{\mu}_{n\,0} = f_n + h_n$ and

$$||f_n||_{L^1(\Omega)} + ||h_n||_{W^{-1,p'}(\Omega)} + ||h_n||_{\mathfrak{M}_b(\Omega)} \leq 5\tilde{\mu}_{n\,0}(\Omega).$$

If we define $F_n = \sum_{k=1}^{n} f_k$ and $G_n = \sum_{k=1}^{n} h_k$, then $\mu_{n\,0} = F_n + G_n$ and

$$||F_n||_{L^1(\Omega)} + ||G_n||_{W^{-1,p'}(\Omega)} + ||G_n||_{\mathfrak{M}_b(\Omega)} \leq 5\tilde{\mu}_0(\Omega).$$

Therefore the convergence statements and (1.3.6) hold. ∎

Proof of Corollary 1.3.6. We set $\nu_n = f_n + \mu_{n,1} - \mu_{n,2} + \vartheta_n$ and $\nu = f + \mu_1 - \mu_2 + \vartheta$. From Lemma 1.3.8 we can write

$$\nu_n = f_n + F_{1\,n} - F_{2\,n} + G_{1\,n} - G_{2\,n} + \mu_{1\,n\,s} - \mu_{2\,n\,s} + \vartheta_n,$$

and

$$\nu = f + F_1 - F_2 + G_1 - G_2 + \mu_{1\,s} - \mu_{2\,s} + \vartheta,$$

and the convergence properties listed in the lemma hold. Therefore we can apply Theorem 1.3.4 and the conclusion follows. ∎

In the next result we prove the main pointwise estimates on renormalized solutions.

Theorem 1.3.9 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $c > 0$, dependent on p and N such that if $\mu \in \mathfrak{M}_b(\Omega)$ and u is a renormalized solution of problem (1.3.2) there holds*

$$-c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^-] \le u \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^+] \quad a.e.\ in\ \Omega. \tag{1.3.7}$$

Proof. We claim the there exist renormalized solutions u_1 and u_2 of problem (1.3.2) with respective data μ^+ and μ^- such that

$$-u_2 \le u \le u_1 \qquad \text{a.e. in } \Omega. \tag{1.3.8}$$

We use the decomposition $\mu = \mu^+ - \mu^- = (\mu_0^+ - \mu_s^+) - (\mu_0^- - \mu_s^-)$. We put $u_k = T_k(u)$, $\mu_k = \chi_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-$, $v_k = \chi_{\{|u|<k\}}\mu_0^+ + \lambda_k^+$. Since $\mu_k \in \mathfrak{M}_0(\Omega)$, problem (1.3.2) with data μ_k admits a unique renormalized solution (see [6]), and clearly u_k is such a solution. Since $v_k \in \mathfrak{M}_0(\Omega)$, problem (1.3.2) with data v_k admits a unique solution $u_{k,1}$ which is furthermore nonnegative and dominates u_k a.e. in Ω. From Corollary 1.3.6, $\{u_{k,1}\}$ converges a.e. in Ω to a renormalized solution u_1 of (1.3.2) with data μ^+ and $u \le u_1$. Similarly $-u \le u_2$ where u_2 is a renormalized solution of (1.3.2) with μ^-. Finally, from [13, Th 6.9] there is a positive constant c dependent only on p and N such that

$$u_1(x) \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^+] \quad \text{and} \quad u_2(x) \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^-] \quad \text{a.e. in } \Omega.$$

This implies the claim. ∎

1.4 Equations with absorption terms

1.4.1 The general case

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that the map $s \mapsto g(x,s)$ is nondecreasing and odd for almost all $x \in \Omega$. If U is a function defined in Ω we define the function $g \circ U$ in Ω by

$$g \circ U(x) = g(x, U(x)) \quad \text{for almost all } x \in \Omega.$$

We consider the problem

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{1.4.1}$$

where $\mu \in \mathfrak{M}_b(\Omega)$. We say that u is a *renormalized solution* of problem (1.4.1) if $g \circ u \in L^1(\Omega)$ and u is a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu - g \circ u \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{1.4.2}$$

Theorem 1.4.1 *Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, such that there exists a nondecreasing sequences $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$, with compact support in Ω, converging to μ_i and $g \circ \big(c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_{i,n}]\big) \in$*

$L^1(\Omega)$ with the same constant c as in Theorem 1.3.9. Then there exists a renormalized solution of

$$\begin{aligned} -\Delta_p u + g\circ u &= \mu_1 - \mu_2 \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{in } \partial\Omega, \end{aligned} \tag{1.4.3}$$

such that

$$-c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_2](x) \le u(x) \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_1](x) \quad a.e.\ in\ \Omega. \tag{1.4.4}$$

Lemma 1.4.2 *Assume g belongs to $L^\infty(\Omega\times\mathbb{R})$, besides the assumptions of Theorem 1.4.1. Let $\lambda_i \in \mathfrak{M}_b^+(\Omega)$ $(i = 1, 2)$, with compact support in Ω. Then there exist renormalized solutions u, u_i, v_i $(i = 1, 2)$ to problems*

$$\begin{aligned} -\Delta_p u + g\circ u &= \lambda_1 - \lambda_2 \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{in } \partial\Omega, \end{aligned} \tag{1.4.5}$$

$$\begin{aligned} -\Delta_p u_i + g\circ u_i &= \lambda_i \quad \text{in } \Omega, \\ u_i &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.4.6}$$

$$\begin{aligned} -\Delta_p v_i &= \lambda_i \quad \text{in } \Omega, \\ v_i &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.4.7}$$

such that

$$\begin{aligned} -c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) &\le -v_2(x) \le -u_2(x) \le u(x) \\ &\le u_1(x) \le v_1(x) \le c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x) \end{aligned} \tag{1.4.8}$$

for a.e $x \in \Omega$.

Proof. Let $\{\rho_n\}$ be a sequence of mollifiers, $\lambda_{i,n} = \rho_n * \lambda_i$, $(i = 1, 2)$ and $\lambda_n = \lambda_{1,n} - \lambda_{2,n}$. Then, for n_0 large enough, $\lambda_{1,n}$, $\lambda_{2,n}$ and λ_n are bounded with compact support in Ω for all $n \ge n_0$ and by minimization there exist unique solutions in $W_0^{1,p}(\Omega)$ to problems

$$\begin{aligned} -\Delta_p u_n + g\circ u_n &= \lambda_n \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p u_{i,n} + g\circ u_{i,n} &= \lambda_{i,n} \quad \text{in } \Omega, \\ u_{i,n} &= 0 \qquad \text{in } \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta_p v_{i,n} &= \lambda_{i,n} \quad \text{in } \Omega, \\ v_{i,n} &= 0 \qquad \text{in } \partial\Omega, \end{aligned}$$

and by the maximum principle, they satisfy

$$-v_{2,n}(x) \le -u_{2,n}(x) \le u_n(x) \le u_{1,n}(x) \le v_{1,n}(x), \quad \forall x \in \Omega,\ \forall n \ge n_0. \tag{1.4.9}$$

Since the λ_i are bounded measure and $g \in L^\infty(\Omega\times\mathbb{R})$ the the sequences of measures $\{\lambda_{1,n} - \lambda_{2,n} - g\circ u_n\}$, $\{\lambda_{i,n} - g\circ u_{i,n}\}$ and $\{\lambda_{i,n}\}$ are uniformly bounded in $\mathfrak{M}^b(\Omega)$. Thus, by Theorem 1.3.3 there exists a subsequence, still denoted by the index n such that $\{u_n\}$, $\{u_{i,n}\}$, $\{v_{i,n}\}$ converge a.e. in Ω to functions $\{u\}$, $\{u_i\}$, $\{v_i\}$ $(i = 1, 2)$ when $n \to \infty$. Furthermore $g\circ u_n$ and $g\circ u_{i,n}$ converge in $L^1(\Omega)$ to $g\circ u$ and $g\circ u_i$ respectively. By Corollary 1.3.5, we can assume that $\{u\}$, $\{u_i\}$, $\{v_i\}$ are renormalized solutions of (1.4.5)-(1.4.7), and by theorem 1.3.9, $v_i(x) \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\lambda_i](x)$, a.e. in Ω. Thus we get (1.4.8). ∎

Lemma 1.4.3 *Let g satisfy the assumptions of Theorem 1.4.1 and let $\lambda_i \in \mathfrak{M}_b^+(\Omega)$ ($i = 1,2$), with compact support in Ω such that $g \circ \left(c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_i]\right) \in L^1(\Omega)$, where c is the constant of Theorem 1.4.1. Then there exist renormalized solutions u, u_i of the problems (1.4.5)-(1.4.6) such that*

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) \leq -u_2(x) \leq u(x) \leq u_1(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x) \tag{1.4.10}$$

for a.e $x \in \Omega$. Furthermore, if ω_i, θ_i have the same properties as the λ_i and satisfy $\omega_i \leq \lambda_i \leq \theta_i$, one can find solutions u_{ω_i} and u_{θ_i} of problems (1.4.6) with right-hand respective side ω_i and θ_i, such that $u_{\omega_i} \leq u_i \leq u_{\theta_i}$.

Proof. From Lemma 1.4.2 there exist renormalized solutions u_n, $u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + T_n(g \circ u_n) &= \lambda_1 - \lambda_2 \quad \text{in } \Omega, \\ u_n &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + T_n(g \circ u_{i,n}) &= \lambda_i \quad \text{in } \Omega, \\ u_{i,n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$, and they satisfy

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x)\ \forall\ x \in \Omega. \tag{1.4.11}$$

Since $\int_\Omega |T_n(g \circ u_n)|dx \leq \lambda_1(\Omega) + \lambda_2(\Omega)$ and $\int_\Omega T_n(g \circ u_{i,n})dx \leq \lambda_i(\Omega)$ thus as in Lemma 1.4.2 one can choose a subsequence, still denoted by the index n such that $\{u_n, u_{1,n}, u_{2,n}\}$ converges a.e. in Ω to $\{u, u_1, u_2\}$ for which (1.4.11) is satisfied a.e. in Ω.
Since $g\circ\left(c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_i]\right) \in L^1(\Omega)$ we derive from (1.4.11) and the dominated convergence theorem that $T_n(g \circ u_n) \to g \circ u$ and $T_n(g \circ u_{i,n}) \to g \circ u_i$ in $L^1(\Omega)$. It follows from Theorem 1.3.4 that u and u_i are respective solutions of (1.4.5), (1.4.6). The last statement follows from the same assertion in Lemma 1.4.2. ∎

Proof of Theorem 1.4.1. From Lemma 1.4.3, there exist renormalized solutions u_n, $u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \mu_{1,n} - \mu_{2,n} \quad \text{in } \Omega, \\ u_n &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \mu_{i,n} \quad \text{in } \Omega, \\ u_{i,n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$ such that $\{u_{i,n}\}$ is nonnegative and nondecreasing and they satisfy

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu_1](x) \tag{1.4.12}$$

a.e. in Ω. As in the proof of Lemma 1.4.3, up to the same subsequence, $\{u_{1,n}\}$, $\{u_{2,n}\}$ and $\{u_n\}$ converge to u_1, u_2 and u a.e. in Ω. Since $g \circ u_{i,n}$ are nondecreasing, positive and $\int_\Omega g \circ u_{i,n}dx \leq \mu_{i,n}(\Omega) \leq \mu_i(\Omega)$, it follows from the monotone convergence theorem that $\{g \circ u_{i,n}\}$ converges to $g \circ u_i$ in $L^1(\Omega)$. Finally, since $|g \circ u_n| \leq g \circ u_1 + g \circ u_2$, $\{g \circ u_n\}$ converges to $g \circ u$ in $L^1(\Omega)$ by dominated convergence. Applying Corollary 1.3.6 we conclude that u is a renormalized solution of (1.4.3) and that (1.4.4) holds. ∎

1.4.2 Proofs of Theorem 1.1.1 and Theorem 1.1.2

We are now in situation of proving the two theorems stated in the introduction.
Proof of Theorem 1.1.1. 1. Since μ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},\frac{q}{q+1-p}}$, μ^+ and μ^- share this property. By Theorem 1.2.8 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and which have the property that $\mathbf{W}^R_{1,p}[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta},q}(\mathbb{R}^N)$, for $i = 1, 2$ and all $n \in \mathbb{N}$. Furthermore, with $R = \mathrm{diam}(\Omega)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](x)\right)^q dx &\leq \int_0^\infty \left(\frac{1}{|.|^\beta}\right)^* (t) \left(\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^* (t)\right)^q dt \\ &\leq c_{34} \int_0^\infty \frac{1}{t^{\frac{\beta}{N}}} \left(\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^* (t)\right)^q dt \\ &\leq c_{34} ||\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]||^q_{L^{\frac{Nq}{N-\beta},q}(\mathbb{R}^N)} \\ &< \infty.\end{aligned}$$

Then the result follows from Theorem 1.4.1.
2. Because μ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},1}$, so are μ^+ and μ^-. Applying again Theorem 1.2.8 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and such that $\mathbf{W}^R_{1,p}[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta},1}(\mathbb{R}^N)$. This implies in particular

$$\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](.)\right)^* (t) \leq c_{35} t^{-\frac{N-\beta}{Nq}}, \qquad \forall t > 0,$$

for some $c_{34} > 0$. Therefore, by Theorem 1.2.3

$$\begin{aligned}\int_\Omega \frac{1}{|x|^\beta} g\left(c\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](x)\right) dx &\leq \int_0^{|\Omega|} \left(\frac{1}{|.|^\beta}\right)^* (t) g\left(c\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^* (t)\right) dt \\ &\leq c_{36} \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(c\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^* (t)\right) dt \\ &\leq c_{36} \int_0^{|\Omega|} \frac{1}{t^{\frac{\beta}{N}}} g\left(c_{35} c t^{-\frac{N-\beta}{Nq}}\right) dt \\ &\leq c_{37} \int_a^\infty g(t) t^{-q-1} dt \\ &< \infty,\end{aligned}$$

where $a > 0$ depends on $|\Omega|$, $c_{35}c$, N, β, q. Thus the result follows by Theorem 1.4.1. ∎
Proof of Theorem 1.1.2. Again we take $R = diam\,(\Omega)$. Let $\{\Omega_n\}_{n\in\mathbb{N}_*}$ be an increasing sequence of compact subsets of Ω such that $\cup_n \Omega_n = \Omega$. We define $\mu_{i,n} = T_n(\chi_{\Omega_n} f_i) + \chi_{\Omega_n} \nu_i$ $(i = 1, 2)$. Then $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ are nondecreasing sequences of elements of $\mathfrak{M}^+_b(\Omega)$ with compact support, and they converge to μ^+ and μ^- respectively. Since for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^\lambda \leq c_\varepsilon n^{\frac{\lambda}{p-1}} + (1+\varepsilon) \left(\mathbf{W}^{2R}_{1,p}[\nu_i]\right)^\lambda,$$

a.e. in Ω, it follows

$$\exp\left(\tau\left(c\mathbf{W}_{1,p}^{2R}[\mu_{i,n}]\right)^{\lambda}\right)\leq c_{\varepsilon,n,c}\exp\left(\tau(1+\varepsilon)\left(c\mathbf{W}_{1,p}^{2R}[\nu_i]\right)^{\lambda}\right).$$

If there holds

$$||\mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i]||_{L^\infty(\Omega)}<\left(\frac{p\ln 2}{\lambda(12\lambda c)^{\lambda}}\right)^{\frac{p-1}{\lambda}},$$

we can choose $\varepsilon>0$ small enough so that

$$\lambda(1+\varepsilon)c^{\lambda}<\frac{p\ln 2}{(12\lambda)^{\lambda}||\mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i]||_{L^\infty(\Omega)}^{\frac{\lambda}{p-1}}}.$$

Hence, by Theorem 1.2.5 with $\eta=\frac{(p-1)(\lambda-1)}{\lambda}$, $\exp\left(\tau(1+\varepsilon)\left(c\mathbf{W}_{1,p}^{2R}[\nu_i]\right)^{\lambda}\right)\in L^1(\Omega)$, which implies $\exp\left(\tau\left(c\mathbf{W}_{1,p}^{2R}[\mu_{i,n}]\right)^{\lambda}\right)\in L^1(\Omega)$. We conclude by Theorem 1.4.1. ■

Bibliographie

[1] D. R. Adams and L. I. Hedberg : *Function Spaces and Potential Theory*, Springer, New York, (1996).

[2] P. Baras, M. Pierre : *Singularités éliminables pour des équations semi linéaires*, Ann. Inst. Fourier Grenoble **34**, 185-206 (1984).

[3] Ph. Benilan, H. Brezis : Nonlinear problems related to the Thomas-Fermi equation, unpublished paper, see [7].

[4] M. F. Bidaut-Veron : *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Elect. J. Diff. Equ. Conf. **8**, 23-34 (2002).

[5] M. F. Bidaut-Veron : *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Adv. Nonlinear Stud. **3**, 25-63 (2003).

[6] L. Boccardo, T. Galouet, L. Orsina : *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **13**, 539-555 (1996).

[7] H. Brezis, Some variational problems of the Thomas-Fermi type, in Variational Inequalities, eds. R.W. Cottle, F. Giannessi and J. L. Lions, Wiley, Chichester , 53-73 (1980).

[8] G. Dal Maso : *On the integral representation of certain local functionals*, Ricerche Mat. **32**, 85-113 (1983).

[9] G. Dal Maso, F. Murat, L. Orsina, A. Prignet : *Renormalized solutions of elliptic equations with general measure data*, Ann.Sc. Norm. Sup. Pisa **28**, 741-808 (1999).

[10] D. Feyel, A. de la Pradelle : *Topologies fines et compactifications associées à certains espaces de Dirichlet*, Ann. Inst. Fourier Grenoble **27**, 121-146 (1977).

[11] L. Grafakos : *Classical Fourier Analysis* 2nd ed., Graduate Texts in Math. **249**, Springer-Verlag (2008).

[12] P. Honzik and B. Jaye : *On the good-λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180 (2012).

[13] N. C. Phuc, I. E. Verbitsky : *Quasilinear and Hessian equations of Lane-Emden type*, Ann. Math. **168**, 859-914 (2008).

[14] E. M. Stein : *Singular Integrals and Differentiability of Functions*, Princeton Univ. Press, Princeton N.J. (1971).

[15] B. O. Tureson, *Nonlinear Potential Theory and Sobolev Spaces*, Springer-Verlag (2000).

[16] L. Véron : *Elliptic equations involving measures*, Stationary partial differential equations. Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam, (2004).

[17] L.Véron : *Singularities of solutions of second other Quasilinear Equations*, Pitman Research Notes in Math. Series **353**, Adison Wesley, Longman (1996).

[18] W. Ziemer : *Weakly Differentiable Functions*, Springer-Verlag (1989).

Chapitre 2

Quasilinear and Hessian type equations with exponential reaction and measure data

Abstract [1]

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω or the whole $\mathbb{R}^N$, where μ is a positive Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ. Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ. Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}^R_{\alpha,p}[P(u)] + f$ in $\mathbb{R}^N$, where $0 < R \leq \infty$ and f is a positive integrable function.

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2.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole $\mathbb{R}^N$, $p > 1$ and $k \in \{1, 2, ..., N\}$. We denote by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1\leq j_1<j_2<...<j_k\leq N} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, ..., \lambda_N$ are the eigenvalues of the Hessian matrix D^2u. Let μ be a positive Radon measure in Ω ; our aim is to study the existence of nonnegative solutions to the following boundary value problems if Ω is bounded,

$$\begin{array}{ll} -\Delta_p u = P(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{2.1.1}$$

and

$$\begin{array}{ll} F_k[-u] = P(u) + \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{array} \tag{2.1.2}$$

where P is an exponential function. If $\Omega = \mathbb{R}^N$, we consider the same equations, but the boundary conditions are replaced by $\inf_{\mathbb{R}^N} u = 0$. When $P(r) = r^q$ with $q > p-1$, Phuc and Verbitsky published a seminal article [20] on the solvability of the corresponding problem (2.1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, assuming that Ω is bounded, they proved that if μ has compact support in Ω it is equivalent to solve (2.1.1) with $P(r) = r^q$, or to have

$$\mu(E) \leq c\text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{2.1.3}$$

where c is a suitable positive constant and $\text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B \left(\mathbf{W}^{2R}_{1,p}[\mu_B](x)\right)^q dx \leq C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset, \tag{2.1.4}$$

where $R =$ diam(Ω). Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{array}{ll} -\Delta_p u = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{2.1.5}$$

for positive measures ω. We refer to [4, 5, 6, 7, 9, 13, 23] for the previous studies of these and other related results. Concerning the k-Hessian operator in a bounded $(k-1)$-convex domain Ω, they proved that if μ has compact support and $||\varphi||_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (2.1.2) with $P(r) = r^q$ with $q > k$ admits a nonnegative solution if and only if

$$\mu(E) \leq c\text{Cap}_{\mathbf{G}_{2k}, \frac{q}{q-k}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{2.1.6}$$

or equivalently

$$\int_B \left[\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B\cap \text{supp}\mu \neq \emptyset. \tag{2.1.7}$$

The results concerning the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 28]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [15, 16], Trudinger and Wang [24, 25, 26], and Labutin [18] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolff potentials of suitable order.

For $s > 1$, $0 < \alpha < \frac{N}{s}$, $\eta \ge 0$ and $0 < T \le \infty$, we recall that the *T-truncated Wolff potential* of a positive Radon measure μ is defined in $\mathbb{R}^N$ by

$$\mathbf{W}^T_{\alpha,s}[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}, \tag{2.1.8}$$

the *T-truncated Riesz potential* of a positive Radon measure μ by

$$\mathbf{I}^T_{\alpha}[\mu](x) = \int_0^T \frac{\mu(B_t(x))}{t^{N-\alpha}} \frac{dt}{t}, \tag{2.1.9}$$

and the *T-truncated η-fractional maximal potential* of μ by

$$\mathbf{M}^\eta_{\alpha,T}[\mu](x) = \sup\left\{\frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} : 0 < t \le T\right\}, \tag{2.1.10}$$

where $h_\eta(t) = (-\ln t)^{-\eta}\chi_{(0,2^{-1})}(t) + (\ln 2)^{-\eta}\chi_{[2^{-1},\infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha,T}[\mu]$ the corresponding *T-truncated fractional maximal potential* of μ. We also denote by $\mathbf{W}_{\alpha,s}[\mu]$ (resp $\mathbf{I}_\alpha[\mu]$, $\mathbf{M}^\eta_\alpha[\mu]$) the *∞-truncated Wolff potential (resp Riesz Potential, $\eta-$ fractional maximal potential)* of μ. When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c. For $l \in \mathbb{N}^*$, we define the *l-truncated exponential function*

$$H_l(r) = e^r - \sum_{j=0}^{l-1}\frac{r^j}{j!}, \tag{2.1.11}$$

and for $a > 0$ and $\beta \ge 1$, we set

$$P_{l,a,\beta}(r) = H_l(ar^\beta). \tag{2.1.12}$$

We put

$$Q_p(s) = \begin{cases} \sum\limits_{q=l}^{\infty} \dfrac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \tag{2.1.13}$$

$Q^*_p(r) = \max\{rs - Q_p(s) : s \ge 0\}$ is the complementary function to Q_p, and define the corresponding Bessel and Riesz capacities respectively by

$$\text{Cap}_{\mathbf{G}_{\alpha p},Q^*_p}(E) = \inf\left\{\int_{\mathbb{R}^N} Q^*_p(f)dx : \mathbf{G}_{\alpha p} * f \ge \chi_E, f \ge 0, Q^*_p(f) \in L^1(\mathbb{R}^N)\right\}, \tag{2.1.14}$$

and

$$\mathrm{Cap}_{\mathbf{I}_{\alpha p},Q_p^*}(E)=\inf\left\{\int_{\mathbb{R}^N}Q_p^*(f)dx:\mathbf{I}_{\alpha p}*f\geq\chi_E,f\geq 0,Q_p^*(f)\in L^1(\mathbb{R}^N)\right\},\qquad(2.1.15)$$

where E is a Borel set in $\mathbb{R}^N$, $\mathbf{G}_{\alpha p}(x)=\mathcal{F}^{-1}\left((1+|.|^2)^{-\frac{\alpha p}{2}}\right)(x)$ is the Bessel kernel of order αp and $I_{\alpha p}(x)=(N-\alpha p)^{-1}|x|^{-(N-\alpha p)}$.

The expressions $a\wedge b$ and $a\vee b$ stand for $\min\{a,b\}$ and $\max\{a,b\}$ respectively. We denote by B_r the ball of center 0 and radius $r>0$. Our main results are the following theorems.

Theorem 2.1.1 *Let $1<p<N$, $a>0$, $l\in\mathbb{N}^*$ and $\beta\geq 1$ such that $l\beta>p-1$. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω, there exists $M>0$ depending on N,p,l,a,β and diam (Ω) (the diameter of Ω) such that if*

$$||\mathbf{M}_{p,2\,diam\,(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\leq M,$$

and $\omega=M||\mathbf{M}_{p,2\,diam(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||_{L^\infty(\mathbb{R}^N)}^{-1}+\mu$ *with* $c_p=1\vee 4^{\frac{2-p}{p-1}}$*, then* $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\omega]\right)$ *is integrable in Ω and the following Dirichlet problem*

$$\begin{array}{ll}-\Delta_p u=P_{l,a,\beta}(u)+\mu & in\ \ \Omega,\\ \quad u=0 & on\ \ \partial\Omega,\end{array}\qquad(2.1.16)$$

admits a nonnegative renormalized solution u which satisfies

$$u(x)\leq 2c_pK_1\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\omega](x)\quad\forall x\in\Omega.\qquad(2.1.17)$$

The role of $K_1=K_1(N,p)$ will be made explicit in Theorem 2.3.4.

Conversely, if (2.1.16) admits a nonnegative renormalized solution u and $P_{l,a,\beta}(u)$ is integrable in Ω, then for any compact set $K\subset\Omega$, there exists a positive constant C depending on N,p,l,a,β and $dist(K,\partial\Omega)$ such that

$$\int_E P_{l,a,\beta}(u)dx+\mu(E)\leq C\,Cap_{\mathbf{G}_p,Q_p^*}(E)\quad for\ all\ Borel\ sets\ E\subset K.\qquad(2.1.18)$$

Furthermore, $u\in W_0^{1,p_1}(\Omega)$ for all $1\leq p_1<p$.

When $\Omega=\mathbb{R}^N$, we have a similar result provided μ has compact suppport.

Theorem 2.1.2 *Let $1<p<N$, $a>0$, $l\in\mathbb{N}^*$ and $\beta\geq 1$ such that $l\beta>\frac{N(p-1)}{N-p}$ and $R>0$. If μ is a nonnegative Radon measure in $\mathbb{R}^N$ with $supp(\mu)\subset B_R$ there exists $M>0$ depending on N,p,l,a,β and R such that if*

$$||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\leq M,$$

and $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$, *then* $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}[\omega]\right)$ *is integrable in* $\mathbb{R}^N$ *and the following problem*

$$\begin{aligned} &-\Delta_p u = P_{l,a,\beta}(u) + \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ &\inf_{\mathbb{R}^N} u = 0, \end{aligned} \tag{2.1.19}$$

admits a p-superharmonic solution u which satisfies

$$u(x) \leq 2c_pK_1\mathbf{W}_{1,p}[\omega](x) \quad \forall x \in \mathbb{R}^N, \tag{2.1.20}$$

(c_p and K_1 as in Theorem 2.1.1).

Conversely, if (2.1.19) *has a solution u and* $P_{l,a,\beta}(u)$ *is locally integrable in* $\mathbb{R}^N$*, then there exists a positive constant C depending on* N, p, l, a, β *such that*

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C\,Cap_{\mathbf{I}_p, Q_p^*}(E) \;\; \forall E \subset \mathbb{R}^N, E \text{ Borel.} \tag{2.1.21}$$

Furthermore, $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ *for all* $1 \leq p_1 < p$*.*

Concerning the k-Hessian operator we recall some notions introduced by Trudinger and Wang [24, 25, 26], and we follow their notations. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1}...\lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = \left[D^2u\right]_k,$$

where in general $[A]_k$ denotes the sum of the k-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly (k-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \to [-\infty, \infty)$ which are k-convex, or subharmonic in the Perron sense (see Definition 2.5.1). In this paper we prove the following theorem (in which expression $\mathbb{E}[q]$ is the largest integer less or equal to q)

Theorem 2.1.3 *Let* $k \in \{1, 2, ..., \mathbb{E}[N/2]\}$ *such that* $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ *such that* $l\beta > k$ *and* $a > 0$*. Let* Ω *be a bounded uniformly (k-1)-convex domain in* $\mathbb{R}^N$*. Let* φ *be a nonnegative continuous function on* $\partial\Omega$ *and* $\mu = \mu_1 + f$ *be a nonnegative Radon measure* *where* μ_1 *has compact support in* Ω *and* $f \in L^q(\Omega)$ *for some* $q > \frac{N}{2k}$. *Let* $K_2 = K_2(N,k)$ *be the constant* K_2 *which appears in Theorem 2.5.3. Then, there exist positive constants* b, M_1 *and* M_2 *depending on* N, k, l, a, β *and diam* (Ω) *such that, if* $\max_{\partial\Omega}\varphi \leq M_2$ *and*

$$||\mathbf{M}_{2k,2diam\,(\Omega)}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M_1,$$

then $P_{l,a,\beta}\left(2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}^{2\,diam\,(\Omega)}[\mu] + b\right)$ *is integrable in* Ω *and the following Dirichlet problem*

$$\begin{array}{ll} F_k[-u] = P_{l,a,\beta}(u) + \mu & in\ \ \Omega, \\ u = \varphi & on\ \ \partial\Omega, \end{array} \tag{2.1.22}$$

admits a nonnegative solution u, *continuous near* $\partial\Omega$, *with* $-u \in \Phi^k(\Omega)$ *which satisfies*

$$u(x) \leq 2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}^{2diam\,(\Omega)}[\mu](x) + b \qquad \forall x \in \Omega. \tag{2.1.23}$$

Conversely, if (2.1.22) *admits a nonnegative solution* u, *continuous near* $\partial\Omega$, *such that* $-u \in \Phi^k(\Omega)$ *and* $P_{l,a,\beta}(u)$ *is integrable in* Ω, *then for any compact set* $K \subset \Omega$, *there exists a positive constant* C *depending on* N, k, l, a, β *and* $dist(K, \partial\Omega)$ *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C\,Cap_{\mathbf{G}_{2k},Q^*_{k+1}}(E) \qquad \forall E \subset K, E\ Borel, \tag{2.1.24}$$

where $Q_{k+1}(s)$ *is defined by (2.1.13) with* $p = k+1$, Q^*_{k+1} *is its complementary function and* $Cap_{\mathbf{G}_{2k},Q^*_{k+1}}(E)$ *is defined accordingly by (2.1.14).*

The following extension holds when $\Omega = \mathbb{R}^N$.

Theorem 2.1.4 *Let* $k \in \{1, 2, ..., \mathbb{E}[N/2]\}$ *such that* $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ *such that* $l\beta > \frac{Nk}{N-2k}$ *and* $a > 0$, $R > 0$. *If* μ *is a nonnegative Radon measure in* $\mathbb{R}^N$ *with* $supp(\mu) \subset B_R$ *there exists* $M > 0$ *depending on* N, k, l, a, β *and* R *such that if*

$$||\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

and $\omega = M||\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$, *then* $P_{l,a,\beta}\left(2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega]\right)$ *is integrable in* $\mathbb{R}^N$ *(*K_2 *as in Theorem 2.1.3) and the following Dirichlet problem*

$$\begin{array}{ll} F_k[-u] = P_{l,a,\beta}(u) + \mu & in\ \ \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u = 0, & \end{array} \tag{2.1.25}$$

admits a nonnegative solution u *with* $-u \in \Phi^k(\mathbb{R}^N)$ *which satisfies*

$$u(x) \leq 2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega](x) \qquad \forall x \in \mathbb{R}^N. \tag{2.1.26}$$

Conversely, if (2.1.25) *admits a nonnegative solution* u *with* $-u \in \Phi^k(\mathbb{R}^N)$ *and* $P_{l,a,\beta}(u)$ *locally integrable in* $\mathbb{R}^N$, *then there exists a positive constant* C *depending on* N, k, l, a, β *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C\,Cap_{\mathbf{I}_{2k},Q^*_{k+1}}(E) \qquad \forall E \subset \mathbb{R}^N, E\ Borel. \tag{2.1.27}$$

where $Cap_{\mathbf{I}_{2k},Q^*_{k+1}}(E)$ *is defined accordingly by (2.1.15).*

The four previous theorems are connected to the following results which deals with a class of nonlinear *Wolff integral equations.*

Theorem 2.1.5 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let f be a nonnegative measurable in $\mathbb{R}^N$ with the property that $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$ is locally integrable in $\mathbb{R}^N$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R such that if*

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad and \quad ||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M, \tag{2.1.28}$$

then there exists a nonnegative function u such that $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u) + \mu] + f \quad in \;\; \mathbb{R}^N, \tag{2.1.29}$$

and

$$u \leq F := 2c_p\mathbf{W}_{\alpha,p}^R[\omega_1] + 2c_p\mathbf{W}_{\alpha,p}^R[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1_{loc}(\mathbb{R}^N), \tag{2.1.30}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$ and $\omega_2 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu_1$.

Conversely, if (2.1.29) *admits a nonnegative solution u and $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$, then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ and R such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a+\varepsilon,\beta}(f)dx + \mu(E) \leq C\, Cap_{\mathbf{G}_{\alpha p},Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N,\ E \text{ Borel.} \tag{2.1.31}$$

When $R = \infty$ in the above theorem, we have a similar result provided f and μ have compact support in $\mathbb{R}^N$.

Theorem 2.1.6 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $0 < \alpha p < N$ and $l\beta > \frac{N(p-1)}{N-\alpha p}$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R such that if f is a nonnegative measurable function in $\mathbb{R}^N$ with support in B_R such that $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$ is locally integrable in $\mathbb{R}^N$ and μ is a positive measure in $\mathbb{R}^N$ with support in B_R which verify*

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad and \quad ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M, \tag{2.1.32}$$

then there exists a nonnegative function u such that $P_{l,a,\beta}(u)$ is integrable in $\mathbb{R}^N$ which satisfies

$$u = \mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u) + \mu] + f \quad in \;\; \mathbb{R}^N, \tag{2.1.33}$$

and

$$u \leq F := 2c_p\mathbf{W}_{\alpha,p}[\omega_1] + 2c_p\mathbf{W}_{\alpha,p}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1(\mathbb{R}^N), \tag{2.1.34}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$ and $\omega_2 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu_1$.

Conversely, if (2.1.33) *admits a nonnegative solution* u *such that* $P_{l,a,\beta}(u)$ *is integrable in* $\mathbb{R}^N$*, then there exists a positive constant* C *depending on* $N, \alpha, p, l, a, \beta$ *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}(f)dx + \mu(E) \leq C\, Cap_{\mathbf{I}_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N,\ E \text{ Borel}. \quad (2.1.35)$$

As an application of the Wolff integral equation we can notice that $\alpha = 1$, equation (2.1.33) is equivalent to

$$-\Delta_p(u - f) = P_{l,a,\beta}(u) + \mu \qquad \text{in } \mathbb{R}^N.$$

When $\alpha = \frac{2k}{k+1}$ and $p = k+1$, it is equivalent to

$$F_k[-u + f] = P_{l,a,\beta}(u) + \mu \qquad \text{in } \mathbb{R}^N.$$

If $p = 2$ equation (2.1.33) becomes linear. If we set $\gamma = 2\alpha$, then

$$\begin{aligned}
\mathbf{W}_{\alpha,2}[\omega](x) &= \int_0^\infty \omega(B_t(x)) \frac{dt}{t^{N-\gamma+1}} \\
&= \int_{\mathbb{R}^N} \left(\int_{|x-y|}^\infty \frac{dt}{t^{N-\gamma+1}} \right) d\mu(y) \\
&= \frac{1}{N-\gamma} \int_{\mathbb{R}^N} \frac{d\omega(y)}{|x-y|^{N-\gamma}} \\
&= \mathbf{I}_\gamma * \omega,
\end{aligned}$$

where $\mathbf{I}_\gamma$ is the Riesz kernel of order γ. Thus (2.1.33) is equivalent to

$$(-\Delta)^\alpha (u - f) = P_{l,a,\beta}(u) + \mu \qquad \text{in } \mathbb{R}^N.$$

Remark 2.1.7 *In case* Ω *is a bounded open set, uniformly bounded of sequence* $\{u_n\}$ (2.2.22) *is essential for the existence of solutions of equations* (2.1.16), (2.1.22) *and* (2.1.29). *Moreover, conditions* $l\beta > p - 1$ *in Theorem 2.1.1, 2.1.5 and* $l\beta > k$ *in Theorem 2.1.3 is necessary so as to get* (2.2.22) *from iteration schemes* (2.2.20)*. Besides, in case* $\Omega = \mathbb{R}^N$*, equation* (2.1.19) *in Theorem 2.1.2 (* (2.1.25) *in Theorem 2.1.4,* (2.1.33) *in Theorem 2.1.6 resp.) has nontrivial solution on* $\mathbb{R}^N$ *if and only if* $l\beta > \frac{N(p-1)}{N-p}$ ($l\beta > \frac{Nk}{N-2k}$, $l\beta > \frac{N(p-1)}{N-\alpha p}$ *resp.). In fact, here we only need to consider equation* (2.1.19)*. Assume that* $l\beta \leq \frac{N(p-1)}{N-p}$*, using Holder inequality we have* $P_{l,a,\beta}(u) \geq cu^\gamma$ *where* $p - 1 < \gamma \leq \frac{N(p-1)}{N-p}$*, so we get from Theorem* (2.3.4)*.*

$$u \geq K\mathbf{W}_{1,p}[cu^\gamma + \mu] \quad \text{in } \mathbb{R}^N$$

for some constant K*. Therefore, we can verify that*

$$\int_E u^\gamma dx + \mu(E) \leq C\, Cap_{\mathbf{I}_p, \frac{\gamma}{\gamma-p+1}}(E) \qquad \forall E \subset \mathbb{R}^N,\ E \text{ Borel}.$$

see Theorem 2.2.7, where C *is a constant and* $Cap_{\mathbf{I}_p, \frac{\gamma}{\gamma-p+1}}$ *is a Riesz capacity.*
Since $N \leq \frac{p\gamma}{\gamma-p+1}$ ($\Leftrightarrow p - 1 < \gamma \leq \frac{N(p-1)}{N-p}$)*,* $Cap_{\mathbf{I}_p, \frac{\gamma}{\gamma-p+1}}(E) = 0$ *for all Borel set* E*, see [1]. Immediately, we deduce* $u \equiv 0$ *and* $\mu \equiv 0$*.*

2.2 Estimates on potentials and Wolff integral equations

We denote by $B_r(a)$ the ball of center a and radius $r > 0$, $B_r = B_r(0)$ and by χ_E the characteristic function of a set E. The next estimates are crucial in the sequel.

Theorem 2.2.1 *Let $\alpha > 0$, $p > 1$ such that $0 < \alpha p < N$.*
1. *There exists a positive constant c_1, depending only on N, α, p such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q \geq p-1$, $0 < R \leq \infty$ we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \leq (c_1 q)^q \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.2.1}$$

2. *Let $R > 0$. There exists a positive constant c_2, depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q \geq p-1$ we have*

$$(c_2 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \leq (c_2 q)^q \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.2.2}$$

where $\mathbf{G}_{\alpha p}[\mu] := \mathbf{G}_{\alpha p} * \mu$ denotes the Bessel potential of order αp of μ.

3. *There exists a positive constant c_3, depending only on N, α, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q \geq 1$ we have*

$$c_3^{-q} \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha}[\mu](x)\right)^q dx \leq \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha}[\mu](x)\right)^q dx \leq c_3^q \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha}[\mu](x)\right)^q dx. \tag{2.2.3}$$

Proof. Note that $\mathbf{W}^R_{\frac{\alpha}{2},2}[\mu] = \mathbf{I}^R_{\alpha}[\mu]$. We can find proof of (2.2.3) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_4 > 0$ such that

$$\int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \geq c_4^q \int_{\mathbb{R}^N} \left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{q}{p-1}} dx \;\; \forall q \geq p-1, \; 0 < R \leq \infty \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N). \tag{2.2.4}$$

We recall that $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}^0_{\alpha p,R}[\mu]$ by (2.1.10). Next we show that for all $q \geq p-1$, $0 < R \leq \infty$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ there holds

$$\int_{\mathbb{R}^N} \left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{q}{p-1}} dx \;\geq (c_5 q)^{-q} \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx, \tag{2.2.5}$$

for some positive constant c_5 depending on N, α, p. Indeed, we denote μ_n by $\chi_{B_n}\mu$ for $n \in \mathbb{N}^*$. By [17, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_6 = c_6(N, \alpha, p) > 0$, $a = a(\alpha, p) > 0$ and $\varepsilon_0 = \varepsilon(N, \alpha, p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $0 < R \leq \infty$ and $0 < \varepsilon < \varepsilon_0$, there holds

$$\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| \leq c_6 \exp\left(-a\varepsilon^{-1}\right) \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| + \left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right|.$$

Multiplying by qt^{q-1} and integrating over $(0, \infty)$, we obtain

$$\begin{aligned}\int_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| dt \leq c_6 \exp\left(-a\varepsilon^{-1}\right) &\int_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| dt \\ &+ \int_0^\infty qt^{q-1} \left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right| dt,\end{aligned}$$

which implies

$$\varepsilon^q\left(3^{-q}-c_6\exp\left(-a\varepsilon^{-1}\right)\right)\int_{\mathbb{R}^N}\left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx\leq\int_{\mathbb{R}^N}\left(\mathbf{M}_{\alpha p,R}\mu_n\right)^{\frac{q}{p-1}}dx.$$

We see that $\sup_{0<\varepsilon<\varepsilon_0}\varepsilon^q\left(3^{-q}-c_6\exp\left(-a\varepsilon^{-1}\right)\right)\geq(c_7q)^{-q}$ for some constant c_7 which does not depend on q. Therefore (2.2.5) follows by Fatou's lemma. Hence, it is easy to obtain (2.2.1) from (2.2.4) and (2.2.5). At end, we obtain (2.2.2) from (2.2.1) and (2.2.3). ■

The next result is proved in [8].

Theorem 2.2.2 *Let $\alpha>0$, $p>1$, $0\leq\eta<p-1$, $0<\alpha p<N$ and $L>0$. Set $\delta=\frac{1}{2}\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\alpha p\log(2)$. Then there exists $C(L)>0$, depending on N, α, p, η and L such that for any $R\in(0,\infty]$, $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$, any $a\in\mathbb{R}^N$ and $0<r\leq L$, there holds*

$$\frac{1}{|B_{2r}(a)|}\int_{B_{2r}(a)}\exp\left(\delta\frac{\left(\mathbf{W}^R_{\alpha,p}[\mu_{B_r(a)}](x)\right)^{\frac{p-1}{p-1-\eta}}}{||\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_r(a)}]||^{\frac{1}{p-1-\eta}}_{L^\infty(B_r(a))}}\right)dx\leq C(L),\tag{2.2.6}$$

where $\mu_{B_r(a)}=\chi_{B_r(a)}\mu$. Furthermore, if $\eta=0$, C is independent of L.

Theorem 2.2.3 *Let $\alpha>0$, $p>1$ with $0<\alpha p<N$, $\beta\geq1$ and $R>0$. Assume $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$ satisfies*

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)}\leq1,\tag{2.2.7}$$

and set $\omega=||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}+\mu$. Then there exist positive constants C, δ_0 and c independent on μ such that $\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)$ is locally integrable in $\mathbb{R}^N$,

$$\left\|\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right]\right\|_{L^\infty(\mathbb{R}^N)}\leq C,\tag{2.2.8}$$

and

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right]\leq c\mathbf{W}^R_{\alpha,p}[\omega]\quad\text{in }\mathbb{R}^N.\tag{2.2.9}$$

Proof. Let δ be as in Theorem 2.2.2. From (2.2.7), we have

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\omega]||_{L^\infty(\mathbb{R}^N)}\leq2.$$

Let $x\in\mathbb{R}^N$. Since $\omega(B_t(y))\leq2t^{N-\alpha p}h_{\frac{(p-1)(\beta-1)}{\beta}}(t)$, for all $r\in(0,R)$ and $y\in\mathbb{R}^N$ we have

$$\begin{aligned}\mathbf{W}^R_{\alpha,p}[\omega](y)&=\mathbf{W}^r_{\alpha,p}[\omega](y)+\int_r^R\left(\frac{\omega(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\&\leq\mathbf{W}^r_{\alpha,p}[\omega](y)+2^{\frac{1}{p-1}}\int_{r\wedge2^{-1}}^{2^{-1}}(-\ln t)^{-\frac{\beta-1}{\beta}}\frac{dt}{t}+2^{\frac{1}{p-1}}\int_{2^{-1}}^{R\vee2^{-1}}(-\ln t)^{-\frac{\beta-1}{\beta}}\frac{dt}{t}\\&\leq\mathbf{W}^r_{\alpha,p}[\omega](y)+c_8(-\ln(r\wedge2^{-1}))^{\frac{1}{\beta}}+c_8.\end{aligned}$$

Thus,

$$\left(\mathbf{W}_{\alpha,p}^{R}\left[\omega\right](y)\right)^{\beta} \leq 3^{\beta-1}\left(\mathbf{W}_{\alpha,p}^{r}\left[\omega\right](y)\right)^{\beta} + c_9 \ln\left(\frac{1}{r\wedge 2^{-1}}\right) + c_9. \tag{2.2.10}$$

Let $\theta \in (0, 2^{-\frac{\beta}{p-1}}]$, since $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$ for all $a, b \in \mathbb{R}$, we get from (2.2.10)

$$\begin{aligned}\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}_{\alpha,p}^{R}\left[\omega\right](y)\right)^{\beta}\right) &\leq \exp\left(\delta 2^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{r}\left[\omega\right](y)\right)^{\beta}\right) + c_{10}\exp\left(\theta c_{11}\ln\left(\frac{1}{r\wedge 2^{-1}}\right)\right)\\ &= \exp\left(\delta 2^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{r}\left[\omega\right](y)\right)^{\beta}\right) + c_{10}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}}.\end{aligned} \tag{2.2.11}$$

For $r > 0$, $0 < t \leq r$, $y \in B_r(x)$ there holds $B_t(y) \subset B_{2r}(x)$. Thus, $\mathbf{W}_{\alpha,p}^{r}[\omega] = \mathbf{W}_{\alpha,p}^{r}[\omega_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.2.6) in Theorem 2.2.2 with $\eta = \frac{(p-1)(\beta-1)}{\beta}$ and $L = 2R$ we get

$$\int_{B_r(x)} \exp\left(\delta 2^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{r}\left[\omega\right]\right)^{\gamma}\right) = \int_{B_r(x)} \exp\left(\delta 2^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{r}\left[\omega_{B_{2r}(x)}\right]\right)^{\gamma}\right) \leq c_{12} r^N.$$

Therefore, taking $\theta = 2^{-\frac{\beta}{p-1}} \wedge \frac{\alpha p}{2c_{11}}$, we deduce from (2.2.11)

$$\begin{aligned}\mathbf{W}_{\alpha,p}^{R}\left[\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}_{\alpha,p}^{R}\left[\omega\right]\right)^{\gamma}\right)\right](x) &\leq \int_0^R \left(c_{12} r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\leq \int_0^R \left(c_{12} r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\frac{\alpha p}{2}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\leq c_{14}.\end{aligned}$$

Hence, we get (2.2.8) with $\delta_0 = \left(2^{-\frac{\beta}{p-1}} \wedge \frac{\alpha p}{2c_{11}}\right)\delta 3^{-\beta}$; we also get (2.2.9) since $\mathbf{W}_{\alpha,p}^{R}[\omega] \geq c_{15}$ for some positive constant $c_{15} > 0$. ■

We recall that H_l and $P_{l,a,\beta}$ have been defined in (2.1.11) and (2.1.12).

Theorem 2.2.4 *Let $\alpha > 0$, $p > 1$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $0 < \alpha p < N$, $l\beta > \frac{N(p-1)}{N-\alpha p}$ and $R > 0$. Assume that $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ has support in B_R and verifies*

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq 1, \tag{2.2.12}$$

and set $\omega = ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$. Then there exist $C = C(N, \alpha, p, l, \beta, R) > 0$ and $\delta_1 = \delta_1(N, \alpha, p, l, \beta, R) > 0$ such that $H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^{\beta}\right)$ is integrable in $\mathbb{R}^N$ and

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^{\beta}\right)\right](x) \leq C\mathbf{W}_{\alpha,p}[\omega](x) \quad \forall\, x \in \mathbb{R}^N. \tag{2.2.13}$$

Proof. We have from (2.2.12)

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq 2. \tag{2.2.14}$$

In particular, $\omega(B_R) \leq c_{16}$. Let $\delta_1 > 0$ and $x \in \mathbb{R}^N$ fixed. We split the Wolff potential $\mathbf{W}_{\alpha,p}[\omega]$ into lower and upper parts defined by

$$\mathbf{L}^t_{\alpha,p}[\omega](x) = \int_t^{+\infty} \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r},$$

and

$$\mathbf{W}^t_{\alpha,p}[\omega](x) = \int_0^t \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Using the convexity we have

$$H_l\left(\delta_1 \left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right) \leq H_l\left(\delta_1 2^\beta \left(\mathbf{L}^t_{\alpha,p}[\omega]\right)^\beta\right) + H_l\left(\delta_1 2^\beta \left(\mathbf{W}^t_{\alpha,p}[\omega]\right)^\beta\right).$$

Thus,

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1 \left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\right](x) \leq c_{17} \int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} + c_{17} \int_0^{+\infty} \left(\frac{\omega^2_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t},$$

where $d\omega^1_t = H_l\left(\delta_1 2^\beta \left(\mathbf{L}^t_{\alpha,p}[\omega]\right)^\beta\right) dx$ and $d\omega^2_t = H_l\left(\delta_1 2^\beta \left(\mathbf{W}^t_{\alpha,p}[\omega]\right)^\beta\right) dx$. Inequality (2.2.13) will follows from the two inequalities below,

$$\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{18} \mathbf{W}_{\alpha,p}[\omega](x), \tag{2.2.15}$$

and

$$\omega^2_t(B_t(x)) \leq c_{18}\omega(B_{4t}(x)). \tag{2.2.16}$$

Step 1 : Proof of (2.2.15). Since $B_r(y) \subset B_{2r}(x)$ for $y \in B_t(x)$ and $r \geq t$, there holds

$$\mathbf{L}^t_{\alpha,p}[\omega](y) \leq \int_t^{+\infty} \left(\frac{\omega(B_{2r}(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} = 2^{\frac{N-\alpha p}{p-1}} \mathbf{L}^{2t}_{\alpha,p}[\omega](x).$$

It follows

$$\omega^1_t(B_t(x)) \leq |B_1(0)| t^N H_l\left(\delta_1 c_{19} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^\beta\right).$$

Thus,

$$\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{20} \int_0^\infty A_t(x) dt, \tag{2.2.17}$$

where

$$A_t(x) = \left(t^{\alpha p} H_l\left(\delta_1 c_{19} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}} \frac{1}{t}.$$

Since $H_l(s) \leq s^l \exp(s)$ for all $s \geq 0$,

$$\begin{aligned} A_t(x) &\leq c_{21} \left(t^{\alpha p} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^{l\beta} \exp\left(\delta_1 c_{19} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}} \frac{1}{t} \\ &= c_{21} t^{\frac{\alpha p}{p-1}-1} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^{\frac{l\beta-p+1}{p-1}} \exp\left(\delta_1 c_{22} \left(\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\right)^\beta\right) \mathbf{L}^{2t}_{\alpha,p}[\omega](x). \end{aligned}$$

Now we estimate $\mathbf{L}^{2t}_{\alpha,p}[\omega]$.

Case 1 : $t \in (0,1)$. From (2.2.14) we deduce

$$\begin{aligned}
\mathbf{L}^{2t}_{\alpha,p}[\omega](x) &\leq \int_{t/2}^{1/2}\left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} + \int_{1/2}^{\infty}\left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\leq c_{23}\int_{t/2}^{1/2}(-ln(s))^{-1+\frac{1}{\beta}}\frac{ds}{s} + \int_{1/2}^{\infty}\left(\frac{\omega(B_R)}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s}\\
&\leq c_{24}\left(-\ln(t/2)\right)^{\frac{1}{\beta}},
\end{aligned}$$

which implies

$$\begin{aligned}
A_t(x) &\leq c_{25}t^{\frac{\alpha p}{p-1}-1}\left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}}\exp\left(\delta_1 c_{26}(-\ln(t/2))\right)\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\\
&= c_{27}t^{\frac{\alpha p}{p-1}-1}\left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}}t^{-\delta_1 c_{26}}\mathbf{L}^{2t}_{\alpha,p}[\omega](x).
\end{aligned}$$

We take $\delta_1 \leq \frac{1}{2c_{26}}\left(\frac{\alpha p}{p-1}-1\right)$ and obtain

$$A_t(x) \leq c_{28}\mathbf{L}^{2t}_{\alpha,p}[\omega](x) \quad \forall t \in (0,1). \tag{2.2.18}$$

Case 2 : $t \geq 1$. We have

$$\mathbf{L}^{2t}_{\alpha,p}[\omega](x) \leq \int_{2t}^{\infty}\left(\frac{\omega(B_R)}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} = c_{29}t^{-\frac{N-\alpha p}{p-1}},$$

thus

$$\begin{aligned}
A_t(x) &\leq c_{30}t^{\frac{\alpha p}{p-1}-1}t^{-\frac{(l\beta-p+1)(N-\alpha p)}{(p-1)^2}}\exp\left(\delta_1 c_{31}t^{-\frac{\beta(N-\alpha p)}{p-1}}\right)\mathbf{L}^{2t}_{\alpha,p}[\omega](x)\\
&\leq c_{32}t^{-1-\gamma}\mathbf{L}^{2t}_{\alpha,p}[\omega](x),
\end{aligned}$$

where $\gamma = \frac{1}{p-1}\left(\frac{l\beta(N-\alpha p)}{p-1}-N\right) > 0$.
Therefore, $A_t(x) \leq c_{33}(t\vee 1)^{-1-\gamma}\mathbf{L}^{2t}_{\alpha,p}[\omega](x)$ for all $t > 0$. Therefore, from (2.2.17)

$$\int_0^{+\infty}\left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \leq c_{34}\int_0^{\infty}(t\vee 1)^{-1-\gamma}\mathbf{L}^{2t}_{\alpha,p}[\omega](x)dt.$$

Using Fubini Theorem we get

$$\begin{aligned}
\int_0^{+\infty}\left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} &\leq c_{34}\int_0^{\infty}\int_0^{t/2}(s\vee 1)^{-1-\gamma}ds\left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\
&\leq c_{35}\int_0^{\infty}\left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\
&= c_{35}\mathbf{W}_{\alpha,p}[\mu](x),
\end{aligned}$$

which follows (2.2.15).

Step 2 : Proof of (2.2.16). For $t>0$, $r\le t$ and $y\in B_t(x)$ we have $B_r(y)\subset B_{2t}(x)$, thus

$$\omega_t^2(B_t(x))=\int_{B_t(x)}H_l\left(\delta_1 2^\beta\left(\mathbf{W}^t_{\alpha,p}[\omega_{B_{2t}(x)}](y)\right)^\beta\right)dy.$$

By Theorem 2.2.2 there exists $c_{36}>0$ such that for $0<\delta_1\le c_{36}$, $0<t<2R$, $z\in\mathbb{R}^N$,

$$\int_{B_{4t}(z)}\exp\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega_{B_{2t}(z)}](y)\right)^\beta\right)dy\le c_{37}t^N. \tag{2.2.19}$$

We take $0<\delta_1\le c_{36}$.

Case 1 : $x\in B_R$. If $0<t<2R$, from (2.2.19) we get

$$\omega_t^2(B_t(x))\le c_{37}t^N\le c_{38}\omega(B_{4t}(x)).$$

If $t\ge 2R$, since for any $|y|\ge 2R$,

$$\mathbf{W}_{\alpha,p}[\omega](y)=\int_{|y|/2}^\infty\left(\frac{\omega(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\le c_{39}\int_{|y|/2}^\infty t^{-1-\frac{N-\alpha p}{p-1}}dt\le c_{40}|y|^{-\frac{N-\alpha p}{p-1}},$$

and thanks to (2.2.19) we have

$$\begin{aligned}\omega_t^2(B_t(x))&\le\int_{B_{2R}}\exp\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega_{B_R}](y)\right)^\beta\right)dy+\int_{\mathbb{R}^N\setminus B_{2R}}H_l\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega](y)\right)^\beta\right)dy\\&\le c_{41}R^N+\int_{\mathbb{R}^N\setminus B_{2R}}H_l\left(c_{42}|y|^{-\frac{\beta(N-\alpha p)}{p-1}}\right)dy\\&\le c_{43}+c_{43}\int_{\mathbb{R}^N\setminus B_{2R}}|y|^{-\frac{l\beta(N-\alpha p)}{p-1}}dy=c_{43}+c_{44}R^{N-\frac{l\beta(N-\alpha p)}{p-1}}\\&\le c_{45}|B_{4t}(x)\cap B_R|\le c_{46}\omega(B_{4t}(x)).\end{aligned}$$

From this we also have $H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\in L^1(\mathbb{R}^N)$.

Case 2 : $x\in\mathbb{R}^N\setminus B_R$. If $|x|>R+t$ then $\omega_t^2(B_t(x))=0$. Next we consider the case $R<|x|\le R+t$. If $0<t<2R$, we have $B_{t/2}((R-\frac{t}{2})\frac{x}{|x|})\subset B_{4t}(x)\cap B_R$; thus from (2.2.19) we get

$$\omega_t^2(B_t(x))\le c_{47}t^N=c_{48}\left|B_{t/2}\left((R-\frac{t}{2})\frac{x}{|x|}\right)\right|\le c_{48}\left|B_{4t}(x)\cap B_R\right|\le c_{49}\omega(B_{4t}(x)).$$

If $t\ge 2R$, as in Case 1 we also obtain $\omega_t^2(B_t(x))\le c_{50}\omega(B_{4t}(x))$ since $B_R\subset B_{4t}(x)$. Hence, we get (2.2.16). Therefore, the result follows with $\delta_1=\left(\frac{1}{2c_{26}}\left(\frac{\alpha p}{p-1}-1\right)\right)\wedge c_{36}$. ∎

In the next result we obtain estimate on a sequence of solutions of Wolff integral inequations obtained by induction.

Theorem 2.2.5 *Assume that the assumptions on α, p, l, a, β, ε, f, μ_1 and μ of Theorem 2.1.5 are fulfilled and R,K are positive real numbers. Suppose that $\{u_m\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned}u_{m+1}&\le K\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u_m)+\mu]+f\quad\forall m\in\mathbb{N},\\u_0&\le K\mathbf{W}^R_{\alpha,p}[\mu]+f.\end{aligned}\tag{2.2.20}$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R such that if

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}^R[\omega_1] + 4c_pK\mathbf{W}_{\alpha,p}^R[\omega_2] + f\right) \in L^1_{loc}(\mathbb{R}^N), \tag{2.2.21}$$

and

$$u_m \leq 2c_pK\mathbf{W}_{\alpha,p}^R[\omega_1] + 2c_pK\mathbf{W}_{\alpha,p}^R[\omega_2] + f \quad \forall m \in \mathbb{N}, \tag{2.2.22}$$

where

$$\omega_1 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu, \tag{2.2.23}$$

$$\omega_2 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu_1, \tag{2.2.24}$$

and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$.
Furthermore, if $f \equiv 0$ then (2.2.21) and (2.2.22) are satisfied with $\omega_2 \equiv 0$.

Proof. The proof is based upon Theorems 2.2.3 and 2.2.4. Set $c_{a,\varepsilon} = 2\left(1 - \left(\frac{a}{a+\varepsilon}\right)^{1/\beta}\right)^{-1}$ and $\overline{a} = a\left(4c_{a,\varepsilon}c_pK\right)^\beta$. If $0 < M \leq 1$ we define ω_1 and ω_2 by (2.2.23) and (2.2.24) respectively. We now assume

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M.$$

We prove first that

$$\mathbf{W}_{\alpha,p}^R\left[H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^R[\omega_i]\right)^\beta\right)\right] \leq \mathbf{W}_{\alpha,p}^R[\omega_i] \quad \text{for } \ i = 1, 2. \tag{2.2.25}$$

By Theorem 2.2.3, there exist $c, \delta_0 > 0$ independent on μ such that $\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^R\left[M^{-1}\omega_i\right]\right)^\beta\right)$ is locally integrable in $\mathbb{R}^N$ and

$$\mathbf{W}_{\alpha,p}^R\left[\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^R\left[M^{-1}\omega_i\right]\right)^\beta\right)\right] \leq c\mathbf{W}_{\alpha,p}^R[M^{-1}\omega_i] \quad \text{in } \mathbb{R}^N.$$

Since $\theta^{-l}H_l(s) \leq H_l(\theta^{-1}s)$ for all $s \geq 0$ and $0 < \theta \leq 1$, it follows

$$\begin{aligned}\mathbf{W}_{\alpha,p}^R\left[M^{-\frac{1}{2}\left(\frac{\beta l}{p-1}+1\right)}H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}_{\alpha,p}^R[\omega_i]\right)^\beta\right)\right] &\leq \mathbf{W}_{\alpha,p}^R\left[H_l\left(\delta_0 M^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^R[\omega_i]\right)^\beta\right)\right] \\ &\leq \mathbf{W}_{\alpha,p}^R\left[\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^R[M^{-1}\omega_i]\right)^\beta\right)\right] \\ &\leq cM^{-\frac{1}{p-1}}\mathbf{W}_{\alpha,p}^R[\omega_i].\end{aligned}$$

Hence,

$$\mathbf{W}_{\alpha,p}^R\left[H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}_{\alpha,p}^R[\omega_i]\right)^\beta\right)\right] \leq cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)}\mathbf{W}_{\alpha,p}^R[\omega_i].$$

Therefore (2.2.25) is achieved if we prove

$$\overline{a} \leq \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} \text{ and } cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)} \leq 1,$$

which is equivalent to

$$M \leq \left(\delta_0 \overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}.$$

Thus, we choose $M = 1 \wedge \left(\delta_0 \overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}$; we obtain (2.2.25) and the fact that $H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_i]\right)^\beta\right) \in L^1_{loc}(\mathbb{R}^N)$.

Now, we prove (2.2.22) by induction. Clearly, (2.2.22) holds with $m = 0$. Next we assume that (2.2.22) holds with $m = n$, and we claim that

$$u_{n+1} \leq 2c_p K\mathbf{W}^R_{\alpha,p}[\omega_1] + 2c_p K\mathbf{W}^R_{\alpha,p}[\omega_2] + f. \tag{2.2.26}$$

In fact, since (2.2.22) holds with $m = n$ and $P_{l,a,\beta}$ is convex, we have

$$\begin{aligned}
P_{l,a,\beta}\left(u_n\right) &\leq P_{l,a,\beta}\left(4c_p K\mathbf{W}^R_{\alpha,p}[\omega_1] + 4c_p K\mathbf{W}^R_{\alpha,p}[\omega_2] + f\right) \\
&\leq P_{l,a,\beta}\left(4c_{a,\varepsilon}c_p K\mathbf{W}^R_{\alpha,p}[\omega_1]\right) + P_{l,\varepsilon,a}\left(4c_{a,\varepsilon}c_p K\mathbf{W}^R_{\alpha,p}[\omega_2]\right) + P_{l,a,\beta}\left(\left(1+\frac{\varepsilon}{a}\right)^{1/\beta} f\right) \\
&= H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^\beta\right) + H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^\beta\right) + P_{l,a+\varepsilon,\beta}(f).
\end{aligned}$$

From this we derive (2.2.21). By the definition of u_{n+1} and the sub-additive property of $\mathbf{W}^R_{\alpha,p}[.]$, we obtain

$$\begin{aligned}
u_{n+1} &\leq K\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^\beta\right) + H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^\beta\right) + P_{l,a+\varepsilon,\beta}(f) + \mu\right] + f \\
&\leq c_p K\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^\beta\right)\right] + c_p K\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^\beta\right)\right] \\
&\qquad + c_p K\mathbf{W}^R_{\alpha,p}\left[P_{l,a+\varepsilon,\beta}(f)\right] + c_p K\mathbf{W}^R_{\alpha,p}\left[\mu\right] + f.
\end{aligned}$$

Hence follows (2.2.26) from (2.2.25). This completes the proof of the theorem. ∎

The next result is obtained by an easy adaptation of the proof Theorem 2.2.5.

Theorem 2.2.6 *Assume that the assumptions on α, p, a, l, β, ε, f, μ_1 and μ of Theorem 2.1.6 are fulfilled and R, K are positive real numbers. Suppose that $\{u_m\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned}
u_{m+1} &\leq K\mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\
u_0 &\leq K\mathbf{W}_{\alpha,p}[\mu] + f.
\end{aligned} \tag{2.2.27}$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R such that if

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}[\omega_3]+4c_pK\mathbf{W}_{\alpha,p}[\omega_4]+f\ \right)\in L^1(\mathbb{R}^N),\tag{2.2.28}$$

and

$$u_m\leq 2c_pK\mathbf{W}_{\alpha,p}[\omega_3]+2c_pK\mathbf{W}_{\alpha,p}[\omega_4]+f\quad\forall m\in\mathbb{N},\tag{2.2.29}$$

where

$$\omega_3=M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R}+\mu,\tag{2.2.30}$$

and

$$\omega_4=M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R}+\mu_1.\tag{2.2.31}$$

Furthermore, if $f\equiv 0$ *then* (2.2.28) *and* (2.2.29) *are satisfied with* $\omega_4\equiv 0$.

Let $P\in C(\mathbb{R}^+)$ be a decreasing positive function. The (α,P)-Orlicz-Bessel capacity of a Borel set $E\subset\mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$\mathrm{Cap}_{\mathbf{G}_\alpha,P}(E)=\inf\left\{\int_{\mathbb{R}^N}P(f)dx:\mathbf{G}_\alpha*f\geq\chi_E,f\geq 0,P(f)\in L^1(\mathbb{R}^N)\right\},$$

and the (α,P)-Orlicz-Riesz capacity

$$\mathrm{Cap}_{\mathbf{I}_\alpha,P}(E)=\inf\left\{\int_{\mathbb{R}^N}P(f)dx:\mathbf{I}_\alpha*f\geq\chi_E,f\geq 0,P(f)\in L^1(\mathbb{R}^N)\right\}.$$

Theorem 2.2.7 *Let* $\alpha>0$, $p>1$, $a>0$, $c>0$, $l\in\mathbb{N}^*$ *and* $\beta\geq 1$ *such that* $l\beta>p-1$ *and* $0<\alpha p<N$. *Let* $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$.
1. *Let* $0<R\leq\infty$. *If* u *is a nonnegative Borel function in* $\mathbb{R}^N$ *such that* $P_{l,a,\beta}(u)$ *is locally integrable in* $\mathbb{R}^N$ *and*

$$u(x)\geq c\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)+\mu](x)\qquad\forall x\in\mathbb{R}^N,\tag{2.2.32}$$

then the following statements holds.
(i) If $R<\infty$, *there exists a positive constant* C_1 *depending on* N,α,p,l,a,β,c *and* R *such that*

$$\int_E P_{l,a,\beta}(u)dx+\mu(E)\leq C_1\,Cap_{\mathbf{G}_{\alpha p},Q_p^*}(E)\qquad\forall E\subset\mathbb{R}^N,\ E\ Borel.\tag{2.2.33}$$

(ii) If $R=\infty$, *there exists a positive constant* C_2 *depending on* N,α,p,l,a,β,c *such that*

$$\int_E P_{l,a,\beta}(u)dx+\mu(E)\leq C_2\,Cap_{\mathbf{I}_{\alpha p},Q_p^*}(E)\qquad\forall E\subset\mathbb{R}^N,\ E\ Borel.\tag{2.2.34}$$

2. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$, $\mu\in\mathfrak{M}^+(\Omega)$ *and* $\delta\in(0,1)$. *If* u *is a nonnegative Borel function in* Ω *such that* $P_{l,a,\beta}(u)$ *is locally integrable in* Ω *and*

$$u(x)\geq c\mathbf{W}^{\delta d(x,\partial\Omega)}_{\alpha,p}[P_{l,a,\beta}(u)+\mu](x)\qquad\forall x\in\Omega,\tag{2.2.35}$$

then, for any compact set $K \subset \Omega$, there exists a positive constant C_3 depending on $N, \alpha, p, l, a, \beta, c, \delta$ and $dist(K, \partial\Omega)$ such that

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \le C_3\, Cap_{\mathbf{G}_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel}, \tag{2.2.36}$$

where Q_p^* is the complementary function to Q_p.

Proof. Set $d\omega = P_{l,a,\beta}(u)dx + d\mu$.
1. We have

$$P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega]\right) dx \le d\omega \quad \text{in } \mathbb{R}^N.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{1}{\omega(B_t(x))} \int_{B_t(x)} |f| d\omega.$$

If $E \subset \mathbb{R}^N$ is a Borel set, we have

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega]\right) dx \le \int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega]\right) dx \le c_{51}\omega(E),$$

for some constant c_{51} only depends on N and $\frac{l\beta}{p-1}$. Since $M_\omega \chi_E \le 1$, we derive

$$\begin{aligned}(M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega](x)\right) &\ge P_{l,a,\beta}\left(c\,(M_\omega \chi_E(x))^{\frac{1}{p-1}}\, \mathbf{W}_{\alpha,p}^R[\omega](x)\right) \\ &\ge P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega_E](x)\right),\end{aligned}$$

where $\omega_E = \chi_E \omega$. Thus

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega_E]\right) dx \le c_{51}\omega(E) \qquad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \tag{2.2.37}$$

From (2.2.1), (2.2.2) and (2.2.3) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega_E](x)\right) dx \ge \int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^R[\omega_E](x)\right) dx \ge \int_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R = \infty,$$

where Q_p is defined by (2.1.13) and $c_{52} = (c_2\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \ne 2$, $c_{52} = c_3^{-1} a^{\frac{1}{\beta}} c$ if $p = 2$ (the constants c_2, c_3 defined in (2.2.2) and (2.2.3), depend on R, therefore $c_{52} = c_{52}(r_K)$)

and $c_{53} = (c_1\beta)^{-1}a^{\frac{p-1}{\beta}}c^{p-1}$ if $p \neq 2$, $c_{53} = a^{\frac{1}{\beta}}c$ if $p = 2$. Thus, from (2.2.37) we obtain that for all Borel set $E \subset \mathbb{R}^N$ there holds

$$\int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } R = \infty.$$

We recall that $Q_p^*(s) = \sup_{t>0}\{st - Q_p(t)\}$ satisfies the sub-additivity Δ_2-condition (see Chapter 2 in [19]).
(i) We assume $R < \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $\mathbf{G}_{\alpha p} * f \geq \chi_E$, we have

$$\begin{aligned}\omega(E) &\leq \int_{\mathbb{R}^N} \mathbf{G}_{\alpha p} * f d\omega_E = (2c_{51})^{-1}\int_{\mathbb{R}^N} \left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right)\left(2c_{51}c_{52}^{-1}f\right) dx \\ &\leq (2c_{51})^{-1}\int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right) dx + (2c_{51})^{-1}\int_{\mathbb{R}^N} Q_p^*\left(2c_{51}c_{52}^{-1}f\right) dx \\ &\leq 2^{-1}\omega(E) + c_{54}\int_{\mathbb{R}^N} Q_p^*\left(f\right) dx,\end{aligned}$$

the last inequality following from the Δ_2-condition. Notice that c_{54}, as well as the next constant c_{55}, depends on r_K. Thus,

$$\omega(E) \leq 2c_{54}\int_{\mathbb{R}^N} Q_p^*\left(f\right) dx.$$

Then, we get

$$\omega(E) \leq c_{55}\mathrm{Cap}_{\mathbf{G}_{\alpha p},Q_p^*}(E) \qquad \forall E \subset \mathbb{R}^N, E \text{ Borel}.$$

Which implies (2.2.33).
(ii) We assume $R = \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $\mathbf{I}_{\alpha p} * f \geq \chi_E$, since $\mathbf{I}_{\alpha p} * \omega_E = \mathbf{I}_{\alpha p}[\omega_E]$, as above we have

$$\begin{aligned}\omega(E) &\leq \int_{\mathbb{R}^N} \mathbf{I}_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} \left(\mathbf{I}_{\alpha p} * \omega_E\right) f dx = \int_{\mathbb{R}^N} \mathbf{I}_{\alpha p}\left[\omega_E\right] f dx \\ &\leq 2^{-1}\omega(E) + c_{56}\int_{\mathbb{R}^N} Q_p^*\left(f\right) dx,\end{aligned}$$

Then, it follows (2.2.34).
2. Let $K \subset \Omega$ be compact. Set $r_K = dist(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right) dx \leq d\omega \quad \text{in } \Omega.$$

Thus, for any Borel set $E \subset K$

$$\int_\Omega (M_\omega\chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right) dx \leq \int_\Omega (M_\omega\chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

As above we get

$$\int_\Omega P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \quad \forall E \subset K, E \text{ Borel}. \tag{2.2.38}$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \le r_K/8$, then $B_t(x) \subset \Omega \backslash \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \le d(x, \partial\Omega) + |x - y| < (1 + \delta) d(x, \partial\Omega) < \frac{1}{4} r_K,$$

thus

$$d(y, K) \ge d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4} r_K > \frac{1}{2} r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \ge \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) \qquad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) = 0 \qquad \forall x \in \Omega^c.$$

Hence we obtain from (2.2.38),

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x)\right) dx \le c_{51} \omega(E) \qquad \forall E \subset K,\ E \text{ Borel}. \tag{2.2.39}$$

As above we also obtain

$$\omega(E) \le c_{57} \mathrm{Cap}_{\mathbf{G}_{\alpha p}, Q_p^*}(E) \qquad \forall E \subset K,\ E \text{ Borel},$$

where the positive constant c_{57} depends on r_K. Inequality (2.2.36) follows and this completes the proof of the Theorem. ∎

Proof of Theorem 2.1.5. Consider the sequence $\{u_m\}_{m \ge 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \ge 0.$$

By Theorem 2.2.5, there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R such that if (2.1.28) holds, then $\{u_m\}_{m \ge 0}$ is well defined and (2.2.21) and (2.2.22) are satisfied. It is easy to see that $\{u_m\}$ is nondecreasing. Hence, thanks to the dominated convergence theorem, we obtain that $u(x) = \lim_{m \to \infty} u_m(x)$ is a solution of equation (2.1.29) which satisfies (2.1.30).
Conversely, we obtain (2.1.31) directly from Theorem 2.2.7, Part 1, (i). ∎

Proof of Theorem 2.1.6. The proof is similar to the previous one by using Theorem 2.2.6 and Theorem 2.2.7, Part 1, (ii). ∎

2.3 Quasilinear Dirichlet problems

Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative parts in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c^{\Omega}_{1,p}$-capacity defined on a compact set $K \subset \Omega$ by

$$c^{\Omega}_{1,p}(K) = \inf\left\{\int_{\Omega} |\nabla\varphi|^p dx : \varphi \geq \chi_K, \varphi \in C^{\infty}_c(\Omega)\right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c^{\Omega}_{1,p}$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega)\cap\mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega)\cap\mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - div\ g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega, \mathbb{R}^N)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined in Ω, finite a.e. and such that $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 2.3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.3.1}$$

if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ^+_k and λ^-_k belonging to $\mathfrak{M}^+_b \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu^+_k \rightharpoonup \mu^+_s$, $\mu^-_k \rightharpoonup \lambda^-_s$ in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2}\, \nabla u.\nabla\varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda^+_k - \int_{\Omega} \varphi d\lambda^-_k,$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega)$.

Remark 2.3.2 *We recall that if u is a renormalized solution to problem (2.3.1), then $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$ for all $r > 1$. From this it follows by Hölder's inequality that $u \in W^{1,p_1}_0(\Omega)$ for all $1 \leq p_1 < p$ provided $e^{a|u|} \in L^1(\Omega)$ for some $a > 0$. Furthermore, $u \geq 0$ a.e. in Ω if $\mu \in \mathfrak{M}^+_b(\Omega)$.*

The following general stability result has been proved in [10, Th 4.1].

Theorem 2.3.3 *Let $\mu = \mu_0 + \mu^+_s - \mu^-_s$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ^+_s, μ^-_s belonging to $\mathfrak{M}^+_s(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}^+_b(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ^+_s and $\{\eta_n\}$ to μ^-_s in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (2.3.1) with data μ_n, then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (2.3.1). Furthermore, $T_k(u_n)$ converges to $T_k(u)$ in $W^{1,p}_0(\Omega)$ for any $k > 0$.*

We also recall the following estimate [20, Th 2.1].

Theorem 2.3.4 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $K_1 > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (2.3.1) with data μ, there holds*

$$\frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \le u(x) \le K_1 \mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu](x) \quad \forall x \in \Omega, \tag{2.3.2}$$

where the positive constant K_1 only depends on N, p.

Proof of Theorem 2.1.1. Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \quad \text{in } \Omega, \\ u_{m+1} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing, see [21]. By Theorem 2.3.4 we have

$$\begin{aligned} \chi_\Omega u_0 &\le K_1 \mathbf{W}_{1,p}^R[\mu], \\ \chi_\Omega u_{m+1} &\le K_1 \mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}, \end{aligned}$$

where $R = 2\,diam\,(\Omega)$. Thus, by Theorem 2.2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_pK_1\mathbf{W}_{1,p}^R[\omega]) \in L^1(\Omega)$ and

$$u_m(x) \le 2c_pK_1\mathbf{W}_{1,p}^R[\omega](x) \quad \forall x \in \Omega, m \in \mathbb{N}, \tag{2.3.3}$$

provided that

$$||M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

where $\omega = M||\mathbf{M}_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. This implies that $\{u_m\}$ is well defined and nondecreasing. Thus $\{u_m\}$ converges a.e in Ω to some function u which satisfies (2.1.17) in Ω. Furthermore, we deduce from (2.3.3) and the monotone convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 2.3.3 we obtain that u is a renormalized solution of (2.1.16).
Conversely, assume that (2.1.16) admits a nonnegative renormalized solution u. By Theorem 2.3.4 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Hence, we achieve (2.1.18) from Theorem 2.2.7, Part 2. ∎

Applications. We consider the case $p = 2$, $\beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in $\mathbb{R}^N$, there exists $M > 0$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \qquad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2\,diam\,\Omega} \frac{\omega(B_t(x))}{t^{N-1}} dt = K(N) \int_0^{2\,diam\,(\Omega)} \frac{\mu(B_t(x))}{t^{N-1}} dt + b \qquad \forall x \in \Omega.$$

where $b = 2K(N)M||\mathbf{M}_{2,2\,diam\,(\Omega)}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}|B_1|(diam\,\Omega)^2$. In the case $N = 2$ this result has already been proved by Richard and Véron [22, Prop 2.4].

2.4 p-superharmonic functions and quasilinear equations in $\mathbb{R}^N$

We recall some definitions and properties of p-superharmonic functions.

Definition 2.4.1 *A function u is said to be p-harmonic in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. A function u is called a p-supersolution in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

Definition 2.4.2 *A lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \to (-\infty, \infty]$ is called p-super-
harmonic if u is not identically infinite and if, for all open $D \subset\subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p-harmonic in D, $v \leq u$ on ∂D implies $v \leq u$ in D.*

Let u be a p-superharmonic in $\mathbb{R}^N$. It is well known that $u \wedge k \in W^{1,p}_{loc}(\mathbb{R}^N)$ is a p-supersolution for all $k > 0$ and $u < \infty$ a.e in $\mathbb{R}^N$, thus, u has a gradient (see the previous section). We also have $|\nabla u|^{p-1} \in L^q_{loc}(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1_{loc}(\mathbb{R}^N)$ and $u \in L^s_{loc}(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ (see [14, Theorem 7.46]). In particular, if $e^{a|u|} \in L^1_{loc}(\mathbb{R}^N)$ for some $a > 0$, then $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$ by Hölder's inequality. Thus for any $0 \leq \varphi \in C^1_c(\Omega)$, by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla (u \wedge k)|^{p-2} \nabla (u \wedge k) \nabla \varphi \geq 0.$$

Hence, by the Riesz Representation Theorem we conclude that there is a nonnegative Radon measure denoted by $\mu[u]$, called Riesz measure, such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [27] will be used to prove the existence of p-superharmonic solutions to quasilinear equations.

Theorem 2.4.3 *Suppose that $\{u_n\}$ is a sequence of nonnegative p-superharmonic functions in $\mathbb{R}^N$ that converges a.e to a p-superharmonic function u. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measures.*

The next theorem is proved in [20]

Theorem 2.4.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there exists a nonnegative p-superharmonic function u in $\mathbb{R}^N$ such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{K_1}\mathbf{W}_{1,p}[\mu](x) \le u(x) \le K_1\mathbf{W}_{1,p}[\mu](x), \tag{2.4.1}$$

for all x in $\mathbb{R}^N$, where the constant K_1 is as in Theorem 2.3.4. Furthermore any p-superharmonic function u in $\mathbb{R}^N$, such that $\inf_{\mathbb{R}^N} u = 0$ satisfies (2.4.1) with $\mu = -\Delta_p u$.

Proof of Theorem 2.1.2. Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of p-superharmonic solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned}$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing. By Theorem 2.4.4 we have

$$\begin{aligned} u_0 &\le K_1\mathbf{W}_{1,p}[\mu], \\ u_{m+1} &\le K_1\mathbf{W}_{1,p}[P_{l,a,\beta}(u_m) + \mu] \ \ \forall m \in \mathbb{N}. \end{aligned}$$

Thus, by Theorem 2.2.6 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_pK_1\mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$ and

$$u_m \le 2c_pK_1\mathbf{W}_{1,p}[\omega] \quad \forall m \in \mathbb{N}, \tag{2.4.2}$$

provided that

$$||M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

where $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$. This implies that $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in $\mathbb{R}^N$ to some p-superharmonic function u which satisfies (2.1.20) in $\mathbb{R}^N$. Furthermore, we deduce from (2.4.2) and the monotone convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\mathbb{R}^N)$. Finally, by Theorem 2.4.3 we conclude that u is a p-superharmonic solution of (2.1.19).
Conversely, assume that (2.1.19) admits a nonnegative renormalized solution u. By Theorem 2.4.4 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain (2.1.21) from Theorem 2.2.7, Part 1, (ii). ∎

2.5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole $\mathbb{R}^N$. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1} ... \lambda_{i_k}.$$

We can see that

$$F_k[u] = \left[D^2u\right]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k-th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \ on \ \ \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 2.5.1 *An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega'})$ satisfying $F_k[v] \leq 0$ in Ω', the following implication is true*

$$u \leq v \ on \ \partial\Omega' \quad \Longrightarrow \quad u \leq v \ \ in \ \ \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k-Hessian operators proved in [25] is fundamental in our study.

Theorem 2.5.2 *Let Ω be either a bounded uniformly (k-1)-convex in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

1 *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.*

2 *If $\{u_n\}$ is a sequence of k-convex functions which converges a.e to u, then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.*

As in the case of quasilinear equations with measure data, precise estimates of solutions of k-Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [25, 18, 20].

Theorem 2.5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2, uniformly (k-1)-convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed under the form*

$$\mu = \mu_1 + f$$

where μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$, or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function u such that $-u \in \Phi^k(\Omega)$ which is continuous near $\partial\Omega$ and is a solution of above equation, satisfies

$$\frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\mu] \leq u(x) \leq K_2\left(\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x) + \max_{\partial\Omega}\varphi\right), \tag{2.5.1}$$

where K_2 is a positive constant independent of x, u and Ω.

Theorem 2.5.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu] < \infty$ a.e. Then there exists u, $-u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in $\mathbb{R}^N$ and*

$$\frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \leq u(x) \leq K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x), \tag{2.5.2}$$

for all x in $\mathbb{R}^N$, where the constant K_2 is the one of the previous Theorem. Furthermore, if u is a nonnegative function such that $\inf_{\mathbb{R}^N} u = 0$ and $-u \in \Phi^k(\mathbb{R}^N)$, then (2.5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem 2.1.3. We defined a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu \quad \text{in } \Omega, \\ u_0 &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{2.5.3}$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu \quad \text{in } \Omega, \\ u_{m+1} &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{2.5.4}$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing, see [21]. By Theorem 2.5.3 we have

$$\begin{aligned} \chi_\Omega u_0 &\leq K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b_0, \\ \chi_\Omega u_{m+1} &\leq K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu] + b_0, \end{aligned} \tag{2.5.5}$$

where $b_0 = K_2 \max_{\partial\Omega}\varphi$ and $R = 2\,diam\,(\Omega)$.
Then, by Theorem 2.2.5 with $f = b_0$ and $\varepsilon = a$, there exists $M_1 > 0$ depending on N, k, l, a, β, K_2 and R such that $P_{l,a,\beta}\left(4K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1] + 2g + b_0\right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1](x) + g + b_0 \qquad \forall x \in \Omega,\, \forall m \geq 0, \tag{2.5.6}$$

provided that

$$||M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M_1 \text{ and } ||M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \le M_1,$$

where $\omega_1 = M_1||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$, $\omega_2 = M_1||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + P_{l,2a,\beta}(b_0)$ and $g = 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_2]$.
Since ω_2 is constant, g has the same property and actually $g = K_2(|B_1|\omega_2)^{\frac{1}{k}}R^2$. On the other hand, one can find constants M_2 depending on N, k, l, a, β, R and M_1 such that if $\max_{\partial\Omega}\varphi \le M_2$, then $||M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \le M_1$.
Hence, we deduce from (2.5.6) that $P_{l,a,\beta}\left(2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b\right) \in L^1(\Omega)$ and

$$u_m(x) \le 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu](x) + b \qquad \forall x \in \Omega, \forall m \ge 0, \tag{2.5.7}$$

for some constant b $(= 2g + b_0)$ depending on N, k, l, a, β, R and M_1. Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta}P_{l,a,\beta}(u_m)) + ((1-\chi_{\Omega_\delta})P_{l,a,\beta}(u_m) + f),$$

where $\Omega_\delta = \{x \in \Omega : d(x,\partial\Omega) > \delta\}$ and $\delta > 0$ is small enough and since u_m is continuous near $\partial\Omega$, then ω satisfies the assumptions of the data in Theorem 2.5.3. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. Thus, $\{u_m\}$ converges a.e in Ω to some function u for which (2.1.23) is satisfied in Ω. Furthermore, we deduce from (2.5.7) and the monotone convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 2.5.2, we obtain that u satisfies (2.1.22) and (2.1.23).
Conversely, assume that (2.1.22) admits nonnegative solution u, continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 2.5.3 we have

$$u(x) \ge \frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Using the part 2 of Theorem 2.2.7, we conclude that (2.1.24) holds. ∎

Proof of Theorem 2.1.4. We define a sequence of nonnegative functions u_m with $-u_m \in \Phi^k(\mathbb{R}^N)$, by the following iterative scheme

$$\begin{array}{l} F_k[-u_0] = \mu \quad \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_0 = 0, \end{array} \tag{2.5.8}$$

and, for $m \ge 0$,

$$\begin{array}{l} F_k[-u_{m+1}] = P_{l,a,\beta}(u_m) + \mu \quad \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} = 0. \end{array} \tag{2.5.9}$$

Clearly, we can assume that $\{u_m\}$ is nondecreasing. By Theorem 2.5.4, we have

$$\begin{array}{l} u_0 \le K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu], \\ u_{m+1} \le K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu]. \end{array} \tag{2.5.10}$$

Thus, by Theorem 2.2.6 with $f \equiv 0$, there exists $M > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta}\left(4K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega]\right) \in L^1(\mathbb{R}^N)$,

$$u_m \leq 2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega] \qquad \forall m \geq 0, \tag{2.5.11}$$

provided that $||M_{2k}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M$, where $\omega = M||\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$. Therefore the sequence $\{u_m\}$ is well defined and nondecreasing. By arguing as in the proof of theorem 2.1.3 we obtain that u satisfies (2.1.25) and (2.1.26).
Conversely, assume that (2.1.25) admits a nonnegative solution u and $-u \in \Phi^k(\mathbb{R}^N)$ such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$, then by Theorem 2.5.4 we have

$$u \geq \frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u) + \mu].$$

Using the part 1, (ii) of Theorem 2.2.7, we conclude that (2.1.27) holds. ∎

Bibliographie

[1] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften **31**, Springer-Verlag (1999).

[2] D.R. Adams, M. Pierre, *Capacitary strong type estimates in semilinear problems*, Ann. Inst. Fourier (Grenoble) **41**, 117-135 (1991).

[3] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, 185-212 (1985).

[4] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Arch. Ration. Mech. Anal. **107** , 293-324 (1989).

[5] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, in : Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8**, 23-34 (2002).

[6] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Adv. Nonlinear Stud. **3**, 25-63 (2003).

[7] M. F. Bidaut-Véron, S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84**, 1-49 (2001).

[8] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102**, 315-337 (2014).

[9] I. Birindelli, F. Demengel, *Some Liouville theorems for the p-Laplacian*, in : Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8**, 35-46 (2002).

[10] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa, **28**, 741-808 (1999).

[11] R. Fefferman, *Strong differentiation with respect to measure*, Amer. J. Math **103**, 33-40 (1981).

[12] F. Ferrari, B. Franchi, I. Verbitsky, *Hessian inequality and the fractional Laplacian*, J. Reine Angew. Math. **667**, 133-148 (2012).

[13] N. Grenon. *Existence results for semilinear elliptic equations with small measure data.*, Ann. Inst. H. Poincaré, Anal. Non Lin. **19**, 1-11 (2002).

[14] J. Heinonen, T. KilpelÃďinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[15] T. Kilpelainen, J. Malý, *Degenerate elliptic equation with measure data and nonlinear potentials*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **19**, 591-613 (1992).

[16] T. Kilpelainen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).

[17] P. Honzik, B. Jaye, *On the good-λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180, (2012).

[18] D. Labutin, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111**, 1-49, (2002).

[19] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Textbooks in Pure and Applied Mathematics (1991).

[20] N. C. Phuc, I. E. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type*, Ann. Math. **168**, 859-914 (2008).

[21] N. C. Phuc, I. E. Verbitsky, *Singular quasilinear and Hessian equation and inequalities*, J. Functional Analysis **256**, 1875-1906 (2009).

[22] Y. Richard, L. Véron, *Isotropic singularities of solutions of nonlinear elliptic inequalities*, Ann. I.H.P. Analyse Non Linéaire **6**,37-72 (1989).

[23] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** 79-142, (2002).

[24] N. S. Trudinger and X.J. Wang, *Hessian measures*, Topological Methods in Nonlinear Analysis **10**, 225-239 (1997).

[25] N. S. Trudinger, X.J. Wang, *Hessian measures II*, Annals of Mathematics, **150**, 579-604 (1999).

[26] N. S. Trudinger, X. J. Wang, *Hessian measures III*, Journal of Functional Analysis, **193**, 1-23 (2002).

[27] N. S. Trudinger, X. J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, Amer. J. Math. **124**, 369-410 (2002).

[28] L. Véron, *Elliptic equations involving measures*, in Stationary Partial Differential Equations, vol. I, Handbook of Equations, Elsevier B.V., pp. 593-712 (2004).

Chapitre 3

Stability properties for quasilinear parabolic equations with measure data and applications

Abstract

Let Ω be a bounded domain of $\mathbb{R}^N$ and $Q = \Omega \times (0,T)$. We first study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $p > 1$, $\mu \in \mathfrak{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators $u \longmapsto \mathcal{A}(u) = \mathrm{div}(A(x,t,\nabla u))$.
As an application, we consider perturbed problems *of type*

$$\begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{G}(u)$ may be an absorption or a source term. In the model case $\mathcal{G}(u) = \pm |u|^{q-1} u$ $(q > p-1)$, or $\mathcal{G}$ has an exponential type. We give existence results when q is subcritical, or when the measure μ is good in time and satisfies suitable capacity conditions.

3.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. We denote by $\mathfrak{M}_b(\Omega)$ and $\mathfrak{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively. We are concerned with the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1.1}$$

where $\mu \in \mathfrak{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and A is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for *a.e.* $(x,t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,t,\xi).\xi \geq c_1 |\xi|^p, \qquad |A(x,t,\xi)| \leq a(x,t) + c_2 |\xi|^{p-1}, \qquad c_1, c_2 > 0, a \in L^{p'}(Q), \tag{3.1.2}$$

$$(A(x,t,\xi) - A(x,t,\zeta)).(\xi - \zeta) > 0 \qquad \text{if } \xi \neq \zeta. \tag{3.1.3}$$

This includes the model problem

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1.4}$$

where Δ_p is the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$.
As an application, we consider problems with a nonlinear term of order 0 :

$$\begin{cases} u_t - \operatorname{div}(A(x,\nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1.5}$$

where A is a Caratheodory function on $\Omega \times \mathbb{R}^N$, such that, for *a.e.* $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,\xi).\xi \geq c_1 |\xi|^p, \qquad |A(x,\xi)| \leq c_2 |\xi|^{p-1}, \qquad c_3, c_4 > 0, \tag{3.1.6}$$

$$(A(x,\xi) - A(x,\zeta)).(\xi - \zeta) > 0 \text{ if } \xi \neq \zeta, \tag{3.1.7}$$

and $\mathcal{G}(u)$ may be an absorption or a source term, and possibly depends on $(x,t) \in Q$. The model problem is the case where $\mathcal{G}$ has a power-type $\mathcal{G}(u) = \pm |u|^{q-1} u$ $(q > p-1)$, or an exponential type.

First make a brief survey of the elliptic associated problem :

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu \in \mathfrak{M}_b(\Omega)$ and assumptions (3.1.6), (3.1.7). When $p = 2$, $A(x,\nabla u) = \nabla u$ existence and uniqueness are proved for general elliptic operators by duality methods in [59]. For $p > 2 - 1/N$, the existence of solutions in the sense of distributions is obtained in [23] and [24]. The condition on p ensures that the gradient ∇u is well defined in $(L^1(\Omega))^N$. For general $p > 1$, new classes of solutions are introduced, first when $\mu \in L^1(\Omega)$, such as *entropy solutions*, and *renormalized solutions*, see [13], and also [58], and existence and uniqueness is obtained. For any $\mu \in \mathfrak{M}_b(\Omega)$ the main work is done in [32, Theorems 3.1, 3.2], where not only existence is proved, but also a stability result, fundamental for applications. Uniqueness is still an open problem.

Next we make a brief survey about problem (3.1.1).

The first studies concern the case $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness is obtained by variational methods, see [44]. In the general case $\mu \in \mathfrak{M}_b(Q)$ and $u_0 \in \mathfrak{M}_b(\Omega)$, the pionner results come from [23], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \tag{3.1.8}$$

see also [56, 57, 26]. The approximated solutions of (3.1.1) lie in Marcinkiewicz spaces $u \in L^{p_c,\infty}(Q)$ and $|\nabla u| \in L^{m_c,\infty}(Q)$, where

$$p_c = p - 1 + \frac{p}{N}, \qquad m_c = p - \frac{N}{N+1}. \tag{3.1.9}$$

This condition (3.1.8) ensures that u and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N+1)$. Uniqueness follows in the case $p = 2$, $A(x, t, \nabla u) = \nabla u$ by duality methods, see [48].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of solutions : *entropy solutions*, and *renormalized solutions*, see [19], [55], see also [3] for a semi-group approach.

A larger set of measures is studied in [33]. They use a notion of parabolic capacity introduced in [33] also see [49, 50] that this was initiated and inspired by Pierre in [51], defined by

$$c_p^Q(E) = \inf(\inf_{E\subset U \text{ open}\subset Q}\{||u||_W : u \in W, u \geq \chi_U \quad a.e. \text{ in } Q\}),$$

for any Borel set $E \subset Q$, where

$$X = L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega)),$$
$$W = \left\{z : z \in X, \quad z_t \in X'\right\}, \text{ embedded with the norm } ||u||_W = ||u||_X + ||u_t||_{X'}.$$

Let $\mathfrak{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c_p^Q-capacity :

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

Then existence and uniqueness of renormalized solutions holds for any measure $\mu \in \mathfrak{M}_b(\Omega) \cap \mathfrak{M}_0(Q)$, called regular (or diffuse) and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [34] ; see also [20] for more general equations.

Next consider *any* measure $\mu \in \mathfrak{M}_b(Q)$. Let $\mathfrak{M}_s(Q)$ be the set of all bounded Radon measures on Q with support on a set of zero c_p^Qcapacity, also called *singular*. Let $\mathfrak{M}_b^+(Q), \mathfrak{M}_0^+(Q), \mathfrak{M}_s^+(Q)$ be the positive cones of $\mathfrak{M}_b(Q), \mathfrak{M}_0(Q), \mathfrak{M}_s(Q)$. From [33], μ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \qquad \mu_0 \in \mathfrak{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \qquad \mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(Q), \tag{3.1.10}$$

and $\mu_0 \in \mathfrak{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \qquad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \tag{3.1.11}$$

and we write $\mu_0 = (f, g, h)$. The solutions of (3.1.1) are searched in a renormalized sense linked to this decomposition, introduced in [19, 49]. In the range (3.1.8) the existence of a renormalized solution relative to the decomposition (3.1.11) is proved in [49], using suitable approximations of μ_0 and μ_s. Uniqueness is still open, as well as in the elliptic case.

Next consider the problem (3.1.5). First we consider the case of an *absorption term* : $\mathcal{G}(u)u \geq 0$.

Let us recall the case $p = 2$, $A(x, \nabla u) = \nabla u$ and $\mathcal{G}(u) = |u|^{q-1}u$ $(q > 1)$. The first results concern the case $\mu = 0$ and u_0 is a Dirac mass in Ω, see [28] : existence holds if and only if $q < (N+2)/N$. Then optimal results are given in [7], for any $\mu \in \mathfrak{M}_b(Q)$ and $u_0 \in \mathfrak{M}_b(\Omega)$. Here two capacities are involved : the elliptic Bessel capacity $\mathrm{Cap}_{\mathbf{G}_\alpha,k}$, $(\alpha > 0, k > 1)$ defined, for any Borel set $E \subset \mathbb{R}^N$, by

$$\mathrm{Cap}_{\mathbf{G}_\alpha,k}(E) = \inf\{||\varphi||_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N), \quad \mathbf{G}_\alpha * \varphi \geq \chi_E\},$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α; and a capacity $\mathrm{Cap}_{\mathbf{G},k}$ $(k > 1)$ adapted to the operator of the heat equation of kernel $\mathbf{G}(x,t) = \chi_{(0,\infty)}(t)(4\pi t)^{-N/2}e^{-|x|^2/4t}$: for any Borel set $E \subset \mathbb{R}^{N+1}$,

$$\mathrm{Cap}_{\mathbf{G},k}(E) = \inf\{||\varphi||_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}), \quad \mathbf{G} * \varphi \geq \chi_E\}.$$

From [7], there exists a solution if and only if μ does not charge the sets of zero $\mathrm{Cap}_{\mathbf{G},q'}-$capacity and u_0 does not charge the sets of zero $\mathrm{Cap}_{2/q,q'}-$capacity.

For $p \neq 2$ such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case $\mu = 0$ with Ω bounded, or $\Omega = \mathbb{R}^N$. The case where u_0 is a Dirac mass in Ω is studied in [35, 39] when $p > 2$, and [29] when $p < 2$. Existence and uniqueness hold in the subcritical case $q < p_c$. If $q \geq p_c$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathfrak{M}_b^+(\Omega)$, the existence is obtained in the sense of distributions in [61], and for any $u_0 \in \mathfrak{M}_b(\Omega)$ in [16]. The case $\mu \in L^1(Q)$, $u_0 = 0$ is treated in [30], and $\mu \in L^1(Q)$, $u_0 = L^1(\Omega)$ in [4] where $\mathcal{G}$ can be multivalued. The case $\mu \in \mathfrak{M}_0(Q)$ is studied in [50], with a new formulation of the solutions, and existence and uniqueness is obtained for any function $\mathcal{G} \in C(\mathbb{R})$ such that $\mathcal{G}(u)u \geq 0$.

The case of a source term $\mathcal{G}(u) = -u^q$ with $u \geq 0$ has been treated in [6] for $p = 2$, where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general p.

3.2 Main results

In *all the sequel* we suppose that p satisfies (3.1.8). Since $W_0^{1,p}(\Omega) \subset L^2(\Omega)$,

$$X = L^p(0, T; W_0^{1,p}(\Omega)), \qquad X' = L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

We first study problem (3.1.1). In Section 3.3 we give some approximations of $\mu \in \mathfrak{M}_b(Q)$, useful for the applications. In Section 3.4 we recall the definition of renormalized solutions, that we call R-solutions of (3.1.1), relative to the decomposition (3.1.11) of μ_0, and study some of their properties.

Our main result is a *stability theorem* for problem (3.1.1), proved in Section 3.5, extending to the parabolic case the stability result of [32, Theorem 3.4], and improving the result of [49] :

Theorem 3.2.1 *Let $A : Q \times \mathbb{R}^N \to \mathbb{R}$ satisfy (3.1.2) and (3.1.3). Let $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(Q),$$

with $f \in L^1(Q), g \in (L^{p'}(Q))^N$, $h \in X$ and $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathfrak{M}_b(Q),$$

with $f_n \in L^1(Q), g_n \in (L^{p'}(Q))^N, h_n \in X$, and $\rho_n, \eta_n \in \mathfrak{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q), \rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathfrak{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n| (Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in X, $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology of measures ; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$. Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \tag{3.2.1}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$. Then up to a subsequence, $\{u_n\}$ converges a.e. in Q to a R-solution u of (3.1.1), and $\{v_n\}$ converges a.e. in Q to $v = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla v_n\}$ converge respectively to $\nabla u, \nabla v$ a.e. in Q, and $\{T_k(v_n)\}$ converge to $T_k(v)$ strongly in X for any $k > 0$.

In Section 3.6 we give applications to problems of type (3.1.5).

We first give an existence result of subcritical type, valid for any measure $\mu \in \mathfrak{M}_b(Q)$:

Theorem 3.2.2 *Let $A : Q \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy (3.1.2) and (3.1.3) with $a \equiv 0$. Let $(x,t,r) \mapsto \mathcal{G}(x,t,r)$ be a Caratheodory function on $Q \times \mathbb{R}$ and $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in $\mathbb{R}^+$, such that*

$$|\mathcal{G}(x,t,r)| \leq G(|r|) \quad \text{for a.e. } (x,t) \in Q \text{ and any } r \in \mathbb{R}, \tag{3.2.2}$$

$$\int_1^\infty G(s)s^{-1-p_c}ds < \infty. \qquad (3.2.3)$$

(i) Suppose that $\mathcal{G}(x,t,r)r \geq 0$, for a.e. (x,t) in Q and any $r \in \mathbb{R}$. Then, for any $\mu \in \mathfrak{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a R-solution u of problem

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + \mathcal{G}(u) = \mu & in\ Q, \\ u = 0 \quad in\ \partial\Omega \times (0,T), \\ u(0) = u_0 \quad in\ \Omega. \end{cases} \qquad (3.2.4)$$

(ii) Suppose that $\mathcal{G}(x,t,r)r \leq 0$, for a.e. $(x,t) \in Q$ and any $r \in \mathbb{R}$, and $u_0 \geq 0, \mu \geq 0$. There exists $\varepsilon > 0$ such that for any $\lambda > 0$, any $\mu \in \mathfrak{M}_b(Q)$ and $u_0 \in L^1(\Omega)$ with $\lambda + |\mu|(Q) + ||u_0||_{L^1(\Omega)} \leq \varepsilon$, problem

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + \lambda\mathcal{G}(u) = \mu & in\ Q, \\ u = 0 \quad in\ \partial\Omega \times (0,T), \\ u(0) = u_0 \quad in\ \Omega, \end{cases} \qquad (3.2.5)$$

admits a nonnegative R-solution.

In particular if $\mathcal{G}(u) = |u|^{q-1} u$, existence holds for any $0 < q < p_c$, for any measure $\mu \in \mathfrak{M}_b(Q)$, small enough if $\mathcal{G}(u) = -|u|^{q-1} u$. In the supercritical case $q \geq p_c$, the class of "admissible" measures, for which there exist solutions, is not known.

Next we give new results relative to *measures that have a good behaviour in t*, based on recent results of [17] relative to the elliptic case. We recall the notions of (truncated) Wolff potential for any nonnegative measure $\omega \in \mathfrak{M}^+(\mathbb{R}^N)$ any $R > 0$, $x_0 \in \mathbb{R}^N$,

$$\mathbf{W}_{1,p}^R[\omega](x_0) = \int_0^R \left(r^{p-N}\omega(B(x_0,r))\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Any measure $\omega \in \mathfrak{M}_b(\Omega)$ is identified with its extension by 0 to $\mathbb{R}^N$. In case of absorption, we obtain the following :

Theorem 3.2.3 *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy (3.1.6) and (3.1.7). Let $p < N$, $q > p - 1$, $\mu \in \mathfrak{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that*

$$|\mu| \leq \omega \otimes F, \quad with\ \omega \in \mathfrak{M}_b^+(\Omega), F \in L^1((0,T)), F \geq 0, \qquad (3.2.6)$$

and ω does not charge the sets of zero $Cap_{\mathbf{G}_p,\frac{q}{q+1-p}}$-capacity. Then there exists a R- solution u of problem

$$\begin{cases} u_t - div(A(x,\nabla u)) + |u|^{q-1}u = f + \mu & in\ Q, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = u_0 \quad in\ \Omega. \end{cases} \qquad (3.2.7)$$

From [7, Proposition 2.3], a measure $\omega \in \mathcal{M}_b(\Omega)$ does not charge the sets of zero $\mathrm{Cap}_{\mathbf{G}_2,\frac{q}{q-1}}$-capacity if and only if $\omega \otimes \chi_{(0,T)}$ does not charge the sets of zero $\mathrm{Cap}_{2,1,\frac{q}{q-1}}$-capacity .

Therefore, when $A(x, \nabla u) = \nabla u$ and $\mu = \omega \otimes \chi_{(0,T)}$, $u_0 \in L^1(\Omega)$, we find again the existence result of [7]. Besides, in view of [33, Theorem 2.16], there exists data $\mu \in \mathcal{M}_b(Q)$ in Theorem 3.2.3 such that $\mu \notin \mathcal{M}_0(Q)$, thus our result is the first one of existence for non diffuse measure. Otherwise our result can be extended to a more general function $\mathcal{G}$, see Remark 3.6.8. We also consider a source term.

Theorem 3.2.4 *Assume that $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (3.1.6) and (3.1.7). Let $p < N$, $q > p - 1$. Let $\mu \in \mathfrak{M}_b^+(Q)$, and $u_0 \in L^\infty(\Omega), u_0 \geq 0$. Assume that*

$$\mu \leq \omega \otimes \chi_{(0,T)}, \qquad \text{with } \omega \in \mathfrak{M}_b^+(\Omega).$$

Then there exist $\lambda_0 = \lambda_0(N, p, q, c_3, c_4 diam(\Omega))$ and $b_0 = b_0(N, p, q, c_3, c_4, \mathrm{diam}(\Omega))$ such that, if

$$\omega(E) \leq \lambda_0 \, Cap_{\mathbf{G}_p, \frac{q}{q-p+1}}(E), \quad \forall E \text{ compact} \subset \mathbb{R}^N, \qquad ||u_0||_{\infty,\Omega} \leq b_0, \tag{3.2.8}$$

there exists a nonnegative R-solution u of problem

$$\begin{cases} u_t - div(A(x, \nabla u)) = u^q + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{3.2.9}$$

which satisfies, a.e. in Q,

$$u(x, t) \leq C \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega](x) + 2||u_0||_{\infty,\Omega}, \tag{3.2.10}$$

where a constant C depends on N, p and the constants c_3, c_4 in inequalities (3.1.6).

Corresponding results in case where $\mathcal{G}$ has exponential type are given at Theorems 3.6.9 and 3.6.14.

3.3 Approximations of measures

For any open set ϖ of $\mathbb{R}^m$ and $F \in (L^k(\varpi))^\nu$, $k \in [1, \infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k,\varpi} = \|F\|_{(L^k(\varpi))^\nu}$.

We give approximations of nonnegative measures in $\mathfrak{M}_b(Q)$. We recall that any measure $\mu \in \mathfrak{M}_0(Q) \cap \mathfrak{M}_b(Q)$ admits a decomposition under the form $\mu = (f, g, h)$ given by (3.1.11). Conversely, any measure of this form, *such that* $h \in L^\infty(Q)$, lies in $\mathfrak{M}_0(Q)$, see [50, Proposition 3.1].

Proposition 3.3.1 *Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(Q)$ with $\mu_0 \in \mathcal{M}_0^+(Q)$ and $\mu_s \in \mathcal{M}_s^+(Q)$.*
(i) Then, we can find a decomposition $\mu_0 = (f, g, h)$ with $f \in L^1(Q), g \in (L^{p'}(Q))^N, h \in L^p((0, T); W_0^{1,p}(\Omega))$ such that

$$||f||_{1,Q} + \|g\|_{p',Q} + ||h||_X + \mu_s(\Omega) \leq 2\mu(Q) \tag{3.3.1}$$

(ii) Furthermore, there exists sequences of measures $\mu_{0,n} = (f_n, g_n, h_n), \mu_{s,n}$ *such that* $f_n, g_n, h_n \in C_c^\infty(Q)$ *strongly converge to* f, g, h *in* $L^1(Q), (L^{p'}(Q))^N$ *and* $L^p((0,T); W_0^{1,p}(\Omega))$ *respectively, and* $\mu_{s,n} \in (C_c^\infty(Q))^+$ *converges to* μ_s *and* $\mu_n := \mu_{0,n} + \mu_{s,n}$ *converges to* μ *in the narrow topology, and satisfying* $|\mu_n|(Q) \le \mu(Q)$,

$$||f_n||_{1,Q} + \|g_n\|_{p',Q} + ||h_n||_X + \mu_{s,n}(Q) \le 2\mu(Q). \tag{3.3.2}$$

Proof. (i) Step 1. Case where μ has a compact support in Q. By [33], we can find a decomposition $\mu_0 = (f, g, h)$ with f, g, h have a compact support in Q. Let $\{\varphi_n\}$ be sequence of mollifiers in $\mathbb{R}^{N+1}$. Then $\mu_{0,n} = \varphi_n * \mu_0 \in C_c^\infty(Q)$ for n large enough. We see that $\mu_{0,n}(Q) = \mu_0(Q)$ and $\mu_{0,n}$ admits the decomposition $\mu_{0,n} = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$. Since $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, we have for n_0 large enough,

$$||f - f_{n_0}||_{1,Q} + ||g - g_{n_0}||_{p',Q} + ||h - h_{n_0}||_X \le \frac{1}{2}\mu_0(Q).$$

Then we obtain a decomposition $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$, such that

$$||\hat{f}||_{1,Q} + ||\hat{g}||_{p',Q} + ||\hat{h}||_X + \mu_s(Q) \le \frac{3}{2}\mu(Q) \tag{3.3.3}$$

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Set $\tilde{\mu}_0 = \theta_0\mu$, and $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$, for any $n \ge 1$. Since $\tilde{\mu}_n = \tilde{\mu}_{0,n} + \tilde{\mu}_{s,n} \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ has compact support with $\tilde{\mu}_{0,n} \in \mathcal{M}_0(Q), \tilde{\mu}_{s,n} \in \mathcal{M}_s(Q)$, by Step 1, we can find a decomposition $\tilde{\mu}_{0,n} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ such that

$$||\tilde{f}_n||_{1,Q} + \|\tilde{g}_n\|_{p',Q} + ||\tilde{h}_n||_X + \tilde{\mu}_{s,n}(\Omega) \le \frac{3}{2}\tilde{\mu}_n(Q)$$

Let $\overline{f}_n = \sum\limits_{k=0}^{n} \tilde{f}_k$, $\overline{g}_n = \sum\limits_{k=0}^{n} \tilde{g}_k$, $\bar{h}_n = \sum\limits_{k=0}^{n} \tilde{h}_k$ and $\bar{\mu}_{s,n} = \sum_{k=0}^{n} \tilde{\mu}_{s,k}$. Clearly, $\theta_n\mu_0 = (\overline{f}_n, \overline{g}_n, \bar{h}_n)$, $\theta_n\mu_s = \bar{\mu}_{s,n}$ and $\{\overline{f}_n\}, \{\overline{g}_n\}, \{\bar{h}_n\}, \{\bar{\mu}_{s,n}\}$ converge strongly to some f, g, h, and μ_s respectively in $L^1(Q)$,$(L^{p'}(Q))^N$, X and $\mathcal{M}_b^+(Q)$, and

$$||\overline{f}_n||_{1,Q} + ||\overline{g}_n||_{p',Q} + ||\bar{h}_n||_X + \bar{\mu}_{s,n}(Q) \le \frac{3}{2}\mu(Q)$$

Therefore, $\mu_0 = (f, g, h)$, and (3.3.1) holds.

(ii) We take a sequence $\{m_n\}$ in $\mathbb{N}$ such that $f_n = \varphi_{m_n} * \overline{f}_n, g_n = \varphi_{m_n} * \overline{g}_n, h_n = \varphi_{m_n} * \bar{h}_n, \varphi_{m_n} * \bar{\mu}_{s,n} \in (C_c^\infty(Q))^+$, $\int_Q \varphi_{m_n} * \bar{\mu}_{s,n} dxdt = \bar{\mu}_{s,n}(Q)$ and

$$||f_n - \overline{f}_n||_{1,Q} + ||g_n - \overline{g}_n||_{p',Q} + ||h_n - \bar{h}_n||_X \le \frac{1}{n+2}\mu(Q).$$

Let $\mu_{0,n} = \varphi_{m_n} * (\theta_n\mu_0) = (f_n, g_n, h_n)$, $\mu_{s,n} = \varphi_{m_n} * \bar{\mu}_{s,n}$ and $\mu_n = \mu_{0,n} + \mu_{s,n}$. Therefore, $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively. And (3.3.2) holds. Furthermore, $\{\mu_{s,n}\}, \{\mu_n\}$ converge to μ_s, μ in the weak topology of measures, and $\mu_{s,n}(Q) = \int_Q \theta_n d\mu_s, \mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu_s(Q), \mu(Q)$, thus $\{\mu_{s,n}\}, \{\mu_n\}$ converges to μ_s, μ in the narrow topology and $|\mu_n|(Q) \leq \mu(Q)$. ∎

Observe that part (i) of Proposition 3.3.1 was used in [49], even if there was no explicit proof. Otherwise part (ii) is a *key point* for finding applications to the stability Theorem. Note also a very useful consequence for approximations by *nondecreasing* sequences :

Proposition 3.3.2 *Let $\mu \in \mathcal{M}_b^+(Q)$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to μ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q)$, $g_n, g \in (L^{p'}(Q))^N$ and $h_n, h \in X$, $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \qquad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and $\{\mu_{n,s}\}$ converges to μ_s (strongly) in $\mathcal{M}_b(Q)$ and

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X + \mu_{n,s}(\Omega) \leq 2\mu(Q). \tag{3.3.4}$$

Proof. Since $\{\mu_n\}$ is nondecreasing, then $\{\mu_{n,0}\}$, $\{\mu_{n,s}\}$ are nondecreasing too. Clearly, $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$. Hence, $\{\mu_{n,s}\}$ converges to μ_s and $\{\mu_{n,0}\}$ converges to μ_0 (strongly) in $\mathcal{M}_b(Q)$. Set $\widetilde{\mu}_{0,0} = \mu_{0,0}$, and $\widetilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for any $n \geq 1$. By Proposition 3.3.1, (i), we can find $\tilde{f}_n \in L^1(Q)$, $\tilde{g}_n \in (L^{p'}(Q))^N$ and $\tilde{h}_n \in X$ such that $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ and

$$||\tilde{f}_n||_{1,Q} + ||\tilde{g}_n||_{p',Q} + ||\tilde{h}_n||_X \leq 2\tilde{\mu}_{n,0}(Q)$$

Let $f_n = \sum_{k=0}^{n} \tilde{f}_k$, $G_n = \sum_{k=0}^{n} \tilde{g}_k$ and $h_n = \sum_{k=0}^{n} \tilde{h}_k$. Clearly, $\mu_{n,0} = (f_n, g_n, h_n)$ and the convergence properties hold with (3.3.4), since

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X \leq 2\mu_0(Q).$$

∎

3.4 Renormalized solutions

3.4.1 Notations and Definition

For any function $f \in L^1(Q)$, we write $\int_Q f$ instead of $\int_Q f dxdt$, and for any measurable set $E \subset Q$, $\int_E f$ instead of $\int_E f dxdt$.

We set $T_k(r) = \max\{\min\{r, k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. We recall that if u is a measurable function defined and finite *a.e.* in Q, such that $T_k(u) \in X$ for any $k > 0$, there exists a measurable function w from Q into $\mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} w$, *a.e.* in Q, and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(Q)$, and (f, g, h) be a decomposition of μ_0 given by (3.1.11), and $\widehat{\mu_0} = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu_0} \notin \mathfrak{M}(Q)$, but we write, for convenience,

$$\int_Q w d\widehat{\mu_0} := \int_Q (fw + g.\nabla w), \qquad \forall w \in X \cap L^\infty(Q).$$

Definition 3.4.1 *Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(Q)$. A measurable function u is a **renormalized solution,** called **R-solution** of (3.1.1) if there exists a decompostion (f, g, h) of μ_0 such that*

$$v = u - h \in L^\sigma(0,T; W_0^{1,\sigma}(\Omega)) \cap L^\infty(0,T; L^1(\Omega)), \quad \forall \sigma \in [1, m_c)\,; \qquad T_k(v) \in X, \quad \forall k > 0, \tag{3.4.1}$$

and :

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$-\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) \\ + \int_Q S'(v)A(x,t,\nabla u).\nabla\varphi + \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v = \int_Q S'(v)\varphi d\widehat{\mu_0}, \tag{3.4.2}$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(.,T) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m\to\infty} \frac{1}{m} \int_{\{m \le v < 2m\}} \phi A(x,t,\nabla u).\nabla v = \int_Q \phi d\mu_s^+, \tag{3.4.3}$$

$$\lim_{m\to\infty} \frac{1}{m} \int_{\{-m \ge v > -2m\}} \phi A(x,t,\nabla u).\nabla v = \int_Q \phi d\mu_s^-. \tag{3.4.4}$$

Remark 3.4.2 *As a consequence, $S(v) \in C([0,T]; L^1(\Omega))$ and $S(v)(.,0) = S(u_0)$ in Ω; and u satisfies the equation*

$$(S(v))_t - \operatorname{div}(S'(v)A(x,t,\nabla u)) + S''(v)A(x,t,\nabla u).\nabla v = fS'(v) - \operatorname{div}(gS'(v)) + S''(v)g.\nabla v, \tag{3.4.5}$$

in the sense of distributions in Q, see [49, Remark 3]. Moreover

$$\|S(v)_t\|_{X'+L^1(Q)} \le \left\|\operatorname{div}(S'(v)A(x,t,\nabla u))\right\|_{X'} + \left\|S''(v)A(x,t,\nabla u).\nabla v\right\|_{1,Q} \\ + \left\|S'(v)f\right\|_{1,Q} + \left\|g.S''(v)\nabla v\right\|_{1,Q} + \left\|\operatorname{div}(S'(v)g)\right\|_{X'}.$$

Thus, if $[-M, M] \supset suppS'$,

$$\|S(v)_t\|_{X'+L^1(Q)} \le C\,\|S\|_{W^{2,\infty}(\mathbb{R})} \,(\, \left\||\nabla u|^p \chi_{|v|\le M}\right\|_{1,Q}^{1/p'} + \left\||\nabla u|^p \chi_{|v|\le M}\right\|_{1,Q} + \||\nabla T_M(v)|\|_{p,Q}^p \\ + \|a\|_{p',Q} + \|a\|_{p',Q}^{p'} + \|f\|_{1,Q} + \|g\|_{p',Q} \left\||\nabla u|^p \chi_{|v|\le M}\right\|_{1,Q}^{1/p} + \|g\|_{p',Q}\,) \tag{3.4.6}$$

We also deduce that, for any $\varphi \in X \cap L^\infty(Q)$, such that $\varphi_t \in X' + L^1(Q)$,

$$\int_\Omega S(v(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x,t,\nabla u).\nabla\varphi \\ + \int_Q S''(v)A(x,t,\nabla u).\nabla v\varphi = \int_Q S'(v)\varphi d\widehat{\mu_0}. \tag{3.4.7}$$

Remark 3.4.3 *Let* u, v *satisfy* (3.4.1). *It is easy to see that the condition (3.4.3) (resp. (3.4.4)) is equivalent to*

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\leq v<2m\}}\phi A(x,t,\nabla u).\nabla u=\int_Q \phi d\mu_s^+ \tag{3.4.8}$$

resp.

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\geq v>-2m\}}\phi A(x,t,\nabla u).\nabla u=\int_Q \phi d\mu_s^-. \tag{3.4.9}$$

In particular, for any $\varphi\in L^{p'}(Q)$ *there holds*

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{m\leq|v|<2m}|\nabla u|\varphi=0, \qquad \lim_{m\to\infty}\frac{1}{m}\int\limits_{m\leq|v|<2m}|\nabla v|\varphi=0. \tag{3.4.10}$$

Remark 3.4.4 *(i) Any function* $U\in X$ *such that* $U_t\in X'+L^1(Q)$ *admits a unique* c_p^Q*-quasi continuous representative, defined* c_p^Q*-quasi a.e. in* Q*, still denoted* U*. Furthermore, if* $U\in L^\infty(Q)$*, then for any* $\mu_0\in\mathfrak{M}_0(Q)$*, there holds* $U\in L^\infty(Q,d\mu_0)$*, see [49, Theorem 3 and Corollary 1].*

(ii) Let u *be any R- solution of problem (3.1.1). Then,* $v=u-h$ *admits a* c_p^Q*-quasi continuous functions representative which is finite* c_p^Q*-quasi a.e. in* Q*, and* u *satisfies definition 3.4.1 for every decomposition* $(\tilde f,\tilde g,\tilde h)$ *such that* $h-\tilde h\in L^\infty(Q)$*, see [49, Proposition 3 and Theorem 4].*

3.4.2 Steklov and Landes approximations

A main difficulty for proving Theorem 3.2.1 is the choice of admissible test functions (S,φ) *in (3.4.2), valid for any R-solution.* Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations :

Definition 3.4.5 *Let* $\varepsilon\in(0,T)$ *and* $z\in L^1_{loc}(Q)$*. For any* $l\in(0,\varepsilon)$*, we define the* ***Steklov time-averages*** $[z]_l,[z]_{-l}$ *of* z *by*

$$[z]_l(x,t)=\frac{1}{l}\int\limits_t^{t+l}z(x,s)ds \qquad \text{for a.e. } (x,t)\in\Omega\times(0,T-\varepsilon),$$

$$[z]_{-l}(x,t)=\frac{1}{l}\int\limits_{t-l}^{t}z(x,s)ds \qquad \text{for a.e. } (x,t)\in\Omega\times(\varepsilon,T).$$

The idea to use this approximation for R-solutions can be found in [22]. Recall some properties, see [50]. Let $\varepsilon\in(0,T)$, and $\varphi_1\in C_c^\infty(\overline{\Omega}\times[0,T))$, $\varphi_2\in C_c^\infty(\overline{\Omega}\times(0,T])$ with $\mathrm{Supp}\varphi_1\subset\overline{\Omega}\times[0,T-\varepsilon]$, $\mathrm{Supp}\varphi_2\subset\overline{\Omega}\times[\varepsilon,T]$. We have

(i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.

(ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \to 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in X, and *a.e.* in Q; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.

(iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \to 0$, there exists a subsequence $\{l_\nu\}$ such that $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$ converge to z, c_p^Q-quasi everywhere in Q.

Next we recall the approximation used in several articles [21, 30, 26], first introduced in [41].

Definition 3.4.6 *Let $k > 0$, and $y \in L^\infty(\Omega)$ and $Y \in X$ such that $||y||_{\infty,\Omega} \leq k$ and $||Y||_{\infty,Q} \leq k$. For any $\nu \in \mathbb{N}$, a **Landes-time approximation** $\langle Y \rangle_\nu$ of the function Y is defined as follows :*

$$\langle Y \rangle_\nu(x,t) = \nu \int_0^t Y(x,s) e^{\nu(s-t)} ds + e^{-\nu t} z_\nu(x) \quad \text{for any } (x,t) \in Q$$

where $\{z_\nu\}$ is a sequence of functions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that $||z_\nu||_{\infty,\Omega} \leq k$, $\{z_\nu\}$ converges to y a.e. in Ω, and $\nu^{-1}||z_\nu||^p_{W_0^{1,p}(\Omega)}$ converges to 0.

Therefore, we can verify that $(\langle Y \rangle_\nu)_t \in X$, $\langle Y \rangle_\nu \in X \cap L^\infty(Q)$, $||\langle Y \rangle_\nu||_{\infty,Q} \leq k$ and $\{\langle Y \rangle_\nu\}$ converges to Y strongly in X and *a.e.* in Q. Moreover, $\langle Y \rangle_\nu$ satisfies the equation $(\langle Y \rangle_\nu)_t = \nu (Y - \langle Y \rangle_\nu)$ in the sense of distributions in Q, and $\langle Y \rangle_\nu(0) = z_\nu$ in Ω. In this paper, we only use the **Landes-time approximation** of the function $Y = T_k(U)$, where $y = T_k(u_0)$.

3.4.3 First properties

In the sequel we use the following notations : for any function $J \in W^{1,\infty}(\mathbb{R})$, nondecreasing with $J(0) = 0$, we set

$$\overline{J}(r) = \int_0^r J(\tau) d\tau, \qquad \mathcal{J}(r) = \int_0^r J'(\tau)\tau d\tau. \tag{3.4.11}$$

It is easy to verify that $\mathcal{J}(r) \geq 0$,

$$\mathcal{J}(r) + \overline{J}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geq s\,(J(r) - J(s)) \qquad \forall r, s \in \mathbb{R}. \tag{3.4.12}$$

In particular we define, for any $k > 0$, and any $r \in \mathbb{R}$,

$$\overline{T_k}(r) = \int_0^r T_k(\tau) d\tau, \qquad \mathcal{T}_k(r) = \int_0^r T_k'(\tau)\tau d\tau, \tag{3.4.13}$$

and we use several times a truncature used in [32] :

$$H_m(r) = \chi_{[-m,m]}(r) + \frac{2m - |s|}{m} \chi_{m<|s|\leq 2m}(r), \qquad \overline{H_m}(r) = \int_0^r H_m(\tau) d\tau. \tag{3.4.14}$$

The next Lemma allows to extend the range of the test functions in (3.4.2). Its proof, given in the Appendix, is obtained by Steklov approximation of the solutions.

Lemma 3.4.7 *Let u be a R-solution of problem (3.1.1). Let $J \in W^{1,\infty}(\mathbb{R})$ be nondecreasing with $J(0) = 0$, and $\overline{J}$ defined by (3.4.11). Then,*

$$\begin{aligned} &\int_Q S'(v)A(x,t,\nabla u).\nabla\left(\xi J(S(v))\right) + \int_Q S''(v)A(x,t,\nabla u).\nabla v \xi J(S(v)) \\ &- \int_\Omega \xi(0)J(S(u_0))S(u_0)dx - \int_Q \xi_t \overline{J}(S(v)) \\ &\leq \int_Q S'(v)\xi J(S(v))d\widehat{\mu_0}, \end{aligned} \tag{3.4.15}$$

for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0) = 0$, and for any $\xi \in C^1(Q) \cap W^{1,\infty}(Q), \xi \geq 0$.

Next we give estimates of the function and its gradient, following the first ones of [26], inspired by the estimates of the elliptic case of [13]. In particular we extend the priori estimates of [49, Proposition 4] given for solutions with smooth data ; see also [33, 42].

Proposition 3.4.8 *If u is a R-solution of problem (3.1.1), then there exists $c = c(p)$ such that, for any $k \geq 1$ and $\ell \geq 0$,*

$$\int\limits_{\ell \leq |v| \leq \ell+k} |\nabla u|^p + \int\limits_{\ell \leq |v| \leq \ell+k} |\nabla v|^p \leq ckM \tag{3.4.16}$$

and

$$\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq c(M + |\Omega|), \tag{3.4.17}$$

where

$$M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + ||a||_{p',Q}^{p'}.$$

As a consequence, for any $k \geq 1$,

$$\text{meas}\{|v| > k\} \leq C_1 M_1 k^{-p_c}, \qquad \text{meas}\{|\nabla v| > k\} \leq C_2 M_2 k^{-m_c}, \tag{3.4.18}$$

$$\text{meas}\{|u| > k\} \leq C_3 M_2 k^{-p_c}, \qquad \text{meas}\{|\nabla u| > k\} \leq C_4 M_2 k^{-m_c}, \tag{3.4.19}$$

where $C_i = C_i(N, p, c_1, c_2)$, $i =$ 1-4, and $M_1 = (M + |\Omega|)^{\frac{p}{N}} M$ and $M_2 = M_1 + M$.

Proof. Set for any $r \in \mathbb{R}$, and $m, k, \ell > 0$,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For $m > k + \ell$, we can choose $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$ as test functions in (3.4.15), where $\overline{H_m}$ is defined at (3.4.14) and $\xi \in C^1([0,T])$ with values in $[0,1]$, independent on x. Since $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$ for all $r \in \mathbb{R}$, we obtain

$$\begin{aligned} &- \textstyle\int_\Omega \xi(0)T_{k,\ell}(u_0)\overline{H_m}(u_0)dx - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(v)) \\ &+ \int\limits_{\{\ell \leq |v| < \ell+k\}} \xi A(x,t,\nabla u).\nabla v - \frac{k}{m} \int\limits_{\{m \leq |v| < 2m\}} \xi A(x,t,\nabla u).\nabla v \leq \textstyle\int_Q H_m(v)\xi T_{k,\ell}(v)d\widehat{\mu_0}. \end{aligned}$$

And

$$\int_Q H_m(v)\xi T_{k,\ell}(v)d\widehat{\mu_0} = \int_Q H_m(v)\xi T_{k,\ell}(v)f + \int\limits_{\{\ell \le |v| < \ell+k\}} \xi \nabla v.g - \frac{k}{m} \int\limits_{\{m \le |v| < 2m\}} \xi \nabla v.g.$$

Let $m \to \infty$; then, for any $k \ge 1$, since $v \in L^1(Q)$ and from (3.4.3), (3.4.4), and (3.4.10), we find

$$-\int_Q \xi_t \overline{T_{k,\ell}}(v) + \int\limits_{\{\ell \le |v| < \ell+k\}} \xi A(x,t,\nabla u).\nabla v \le \int\limits_{\{\ell \le |v| < \ell+k\}} \xi \nabla v.g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \tag{3.4.20}$$

Next, we take $\xi \equiv 1$. We verify that there exists $c = c(p)$ such that

$$A(x,t,\nabla u).\nabla v - \nabla v.g \ge \frac{c_1}{4}(|\nabla u|^p + |\nabla v|^p) - c(|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

where c_1 is the constant in (3.1.2). Hence (3.4.16) follows. Thus, from (3.4.20) and the Hölder inequality, we get, with another constant c, for any $\xi \in C^1([0,T])$ with values in $[0,1]$,

$$-\int_Q \xi_t \overline{T_{k,\ell}}(v) \le ckM$$

Thus $\int_\Omega \overline{T_{k,\ell}}(v)(t)dx \le ckM$, for *a.e.* $t \in (0,T)$. We deduce (3.4.17) by taking $k = 1, \ell = 0$, since $\overline{T_{1,0}}(r) = \overline{T_1}(r) \ge |r| - 1$, for any $r \in \mathbb{R}$.

Next, from the Gagliardo-Nirenberg embedding Theorem, we have

$$\int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \le C_1 \|v\|^{\frac{p}{N}}_{L^\infty((0,T);L^1(\Omega))} \int_Q |\nabla T_k(v)|^p,$$

where $C_1 = C_1(N,p)$. Then, from (3.4.16) and (3.4.17), we get, for any $k \ge 1$,

$$\begin{aligned} \operatorname{meas}\{|v| > k\} &\le k^{-\frac{p(N+1)}{N}} \int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \\ &\le C \|v\|^{\frac{p}{N}}_{L^\infty((0,T);L^1(\Omega))} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(v)|^p \\ &\le C_2 M_1 k^{-p_c}, \end{aligned}$$

with $C_2 = C_2(N,p,c_1,c_2)$. We obtain

$$\begin{aligned} \operatorname{meas}\{|\nabla v| > k\} &\le \frac{1}{k^p}\int_0^{k^p} \operatorname{meas}(\{|\nabla v|^p > s\})\, ds \\ &\le \operatorname{meas}\left\{|v| > k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p}\int_0^{k^p} \operatorname{meas}\left(\left\{|\nabla v|^p > s, |v| \le k^{\frac{N}{N+1}}\right\}\right) ds \\ &\le C_2 M_1 k^{-m_c} + \frac{1}{k^p} \int\limits_{|v| \le k^{\frac{N}{N+1}}} |\nabla v|^p \le C_2 M_2 k^{-m_c}, \end{aligned}$$

with $C_3 = C_3(N,p,c_1,c_2)$. Furthermore, for any $k \ge 1$,

$$\operatorname{meas}\{|h| > k\} + \operatorname{meas}\{|\nabla h| > k\} \le C_4 k^{-p} \|h\|^p_X,$$

where $C_4 = C_4(N,p,c_1,c_2)$. Therefore, we easily get (3.4.19). ■

Remark 3.4.9 *If $\mu \in L^1(Q)$ and $a \equiv 0$ in (3.1.2), then (3.4.16) holds for all $k > 0$ and the term $|\Omega|$ in inequality (3.4.17) can be removed where $M = ||u_0||_{1,\Omega} + |\mu|(Q)$. Furthermore, (3.4.19) is stated as follows :*

$$\text{meas}\{|u| > k\} \leq C_3 M^{\frac{p+N}{N}} k^{-p_c}, \qquad \text{meas}\{|\nabla u| > k\} \leq C_4 M^{\frac{N+2}{N+1}} k^{-m_c}, \forall k > 0. \quad (3.4.21)$$

To see last inequality, we do in the following way :

$$\begin{aligned}\text{meas}\{|\nabla v| > k\} &\leq \frac{1}{k^p}\int_0^{k^p} \text{meas}\left(\{|\nabla v|^p > s\}\right) ds \\ &\leq \text{meas}\left\{|v| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p}\int_0^{k^p} \text{meas}\left\{|\nabla v|^p > s, |v| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} ds \\ &\leq C_4 M^{\frac{N+2}{N+1}} k^{-m_c}.\end{aligned}$$

Proposition 3.4.10 *Let $\{\mu_n\} \subset \mathfrak{M}_b(Q)$, and $\{u_{0,n}\} \subset L^1(\Omega)$, with*

$$\sup_n |\mu_n|(Q) < \infty, \text{ and } \sup_n ||u_{0,n}||_{1,\Omega} < \infty.$$

Let u_n be a R-solution of (3.1.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ and $u_{0,n}$, relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$, and $v_n = u_n - h_n$. Assume that $\{f_n\}$ is bounded in $L^1(Q)$, $\{g_n\}$ bounded in $(L^{p'}(Q))^N$ and $\{h_n\}$ bounded in X.

Then, up to a subsequence, $\{v_n\}$ converges a.e. to a function v, such that $T_k(v) \in X$ and $v \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0,T); L^1(\Omega))$ for any $\sigma \in [1, m_c)$. And

(i) $\{v_n\}$ converges to v strongly in $L^\sigma(Q)$ for any $\sigma \in [1, m_c)$, and $\sup \|v_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$,

(ii) $\sup_{k>0} \sup_n \frac{1}{k+1}\int_Q |\nabla T_k(v_n)|^p < \infty$,

(iii) $\{T_k(v_n)\}$ converges to $T_k(v)$ wealkly in X, for any $k > 0$,

(iv) $\{A(x, t, \nabla(T_k(v_n) + h_n))\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$.

Proof. Take $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0) = 0$. We combine (3.4.6) with (3.4.16), and deduce that $\{S(v_n)_t\}$ is bounded in $X' + L^1(Q)$ and $\{S(v_n)\}$ bounded in X. Hence, $\{S(v_n)\}$ is relatively compact in $L^1(Q)$. On the other hand, we choose $S = S_k$ such that $S_k(z) = z$, if $|z| < k$ and $S(z) = 2k$ sign$z$, if $|z| > 2k$. Thanks to (3.4.17), we obtain

$$\begin{aligned}\text{meas}\{|v_n - v_m| > \sigma\} &\leq \text{meas}\{|v_n| > k\} + \text{meas}\{|v_m| > k\} + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\} \\ &\leq \frac{1}{k}(\|v_n\|_{1,Q} + \|v_m\|_{1,Q}) + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\} \\ &\leq \frac{C}{k} + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\}. \qquad (3.4.22)\end{aligned}$$

Thus, up to a subsequence $\{u_n\}$ is a Cauchy sequence in measure, and converges *a.e.* in Q to a function u. Thus, $\{T_k(v_n)\}$ converges to $T_k(v)$ weakly in X, since $\sup_n \|T_k(v_n)\|_X < \infty$ for any $k > 0$. And $\left\{|\nabla(T_k(v_n) + h_n)|^{p-2}\nabla(T_k(v_n) + h_n)\right\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$. Furthermore, from (3.4.18), $\{v_n\}$ converges to v strongly in $L^\sigma(Q)$, for any $\sigma < p_c$. ■

3.5 The convergence theorem

We first recall some properties of the measures, see [49, Lemma 5], [32].

Proposition 3.5.1 *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(Q)$, where μ_s^+ and μ_s^- are concentrated, respectively, on two disjoint sets E^+ and E^- of zero c_p^Q-capacity. Then, for any $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leq \delta, \qquad \mu_s^-(E^- \backslash K_\delta^-) \leq \delta,$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$ with values in $[0,1]$, such that $\psi_\delta^+, \psi_\delta^- = 1$ respectively on K_δ^+, K_δ^-, and $supp(\psi_\delta^+) \cap supp(\psi_\delta^-) = \emptyset$, and

$$||\psi_\delta^+||_X + ||(\psi_\delta^+)_t||_{X'+L^1(Q)} \leq \delta, \qquad ||\psi_\delta^-||_X + ||(\psi_\delta^-)_t||_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions $(\psi_\delta^+)_t = \left(\psi_\delta^+\right)_t^1 + \left(\psi_\delta^+\right)_t^2$ and $(\psi_\delta^-)_t = \left(\psi_\delta^-\right)_t^1 + \left(\psi_\delta^-\right)_t^2$ in $X' + L^1(Q)$, such that

$$\left\|\left(\psi_\delta^+\right)_t^1\right\|_{X'} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^+\right)_t^2\right\|_{1,Q} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^1\right\|_{X'} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^2\right\|_{1,Q} \leq \frac{\delta}{3}. \tag{3.5.1}$$

Both $\left\{\psi_\delta^+\right\}$ and $\left\{\psi_\delta^-\right\}$ converge to 0, *-weakly in $L^\infty(Q)$, and strongly in $L^1(Q)$ and up to subsequences, a.e. in Q, as δ tends to 0.

Moreover if ρ_n and η_n are as in Theorem 3.2.1, we have, for any $\delta, \delta_1, \delta_2 > 0$,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n,\delta), \qquad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \qquad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \tag{3.5.2}$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n,\delta_1,\delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \tag{3.5.3}$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n,\delta_1,\delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \tag{3.5.4}$$

Hereafter, if $n, \varepsilon, ..., \nu$ are real numbers, and a function ϕ depends on $n, \varepsilon, ..., \nu$ and eventual other parameters $\alpha, \beta, .., \gamma$, and $n \to n_0, \varepsilon \to \varepsilon_0, .., \nu \to \nu_0$, we write $\phi = \omega(n, \varepsilon, .., \nu)$, then this means $\overline{\lim}_{\nu\to\nu_0} .. \overline{\lim}_{\varepsilon\to\varepsilon_0} \overline{\lim}_{n\to n_0} |\phi| = 0$, when the parameters $\alpha, \beta, .., \gamma$ are fixed. In the same way, $\phi \leq \omega(n, \varepsilon, \delta, ..., \nu)$ means $\overline{\lim}_{\nu\to\nu_0} .. \overline{\lim}_{\varepsilon\to\varepsilon_0} \overline{\lim}_{n\to n_0} \phi \leq 0$, and $\phi \geq \omega(n, \varepsilon, .., \nu)$ means $-\phi \leq \omega(n, \varepsilon, .., \nu)$.

Remark 3.5.2 *In the sequel we use a convergence property, consequence of the Dunford-Pettis theorem, still used in [32] : If $\{a_{1,n}\}$ is a sequence in $L^1(Q)$ converging to a_1 weakly in $L^1(Q)$ and $\{b_{1,n}\}$ a bounded sequence in $L^\infty(Q)$ converging to b_1, a.e. in Q, then*

$$\lim_{n\to\infty} \int_Q a_{1,n} b_{1,n} dxdt = \int_Q a_1 b_1 .dxdt.$$

Next we prove Thorem 3.2.1.

Scheme of the proof. Let $\{\mu_n\}, \{u_{0,n}\}$ and $\{u_n\}$ satisfying the assumptions of Theorem 3.2.1. Then we can apply Proposition 3.4.10. Setting $v_n = u_n - h_n$, up to subsequences, $\{u_n\}$ converges *a.e.* in Q to some function u, and $\{v_n\}$ converges *a.e.* to $v = u - h$, such that $T_k(v) \in X$ and $v \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0,T); L^1(\Omega))$ for every $\sigma \in [1, m_c)$. And $\{v_n\}$ satisfies the conclusions (i) to (iv) of Proposition 3.4.10. We have

$$\begin{aligned}\mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-,\end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \ \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \ \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \tag{3.5.5}$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathfrak{M}_b^+(Q) \cap \mathfrak{M}_0(Q), \quad \text{and} \quad \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}. \tag{3.5.6}$$

Let E^+, E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. For any $\delta_1, \delta_2 > 0$, let $\psi_{\delta_1}^+, \psi_{\delta_2}^+$ and $\psi_{\delta_1}^-, \psi_{\delta_2}^-$ as in Proposition 3.5.1 and set

$$\Phi_{\delta_1,\delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near E

$$I_1 := \int_{\{|v_n| \leq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \left(v_n - \langle T_k(v)\rangle_\nu\right) \leq \omega(n, \nu, \delta_1, \delta_2), \tag{3.5.7}$$

and far from E,

$$I_2 := \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1,\delta_2}) A(x,t,\nabla u_n).\nabla (v_n - \langle T_k(v)\rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2). \tag{3.5.8}$$

Then it follows that

$$\overline{\lim}_{n,\nu} \int_{\{|v_n| \leq k\}} A(x,t,\nabla u_n).\nabla \left(v_n - \langle T_k(v)\rangle_\nu\right) \leq 0, \tag{3.5.9}$$

which implies

$$\overline{\lim}_{n\to\infty} \int_{\{|v_n| \leq k\}} A(x,t,\nabla u_n).\nabla \left(v_n - T_k(v)\right) \leq 0, \tag{3.5.10}$$

since $\{\langle T_k(v)\rangle_\nu\}$ converges to $T_k(v)$ in X. On the other hand, from the weak convergence of $\{T_k(v_n)\}$ to $T_k(v)$ in X, we verify that

$$\int_{\{|v_n| \leq k\}} A(x,t,\nabla (T_k(v) + h_n)).\nabla \left(T_k(v_n) - T_k(v)\right) = \omega(n).$$

Thus we get

$$\int_{\{|v_n|\leq k\}} \left(A(x,t,\nabla u_n) - A(x,t,\nabla(T_k(v)+h_n))\right).\nabla\left(u_n - (T_k(v)+h_n)\right) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \qquad a.e. \text{ in } Q. \tag{3.5.11}$$

Therefore, $\{A(x,t,\nabla u_n)\}$ converges to $A(x,t,\nabla u)$ weakly in $(L^{p'}(Q))^N$; and from (3.5.10) we find

$$\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla u_n).\nabla T_k(v_n) \leq \int_Q A(x,t,\nabla u)\nabla T_k(v).$$

Otherwise, $\{A(x,t,\nabla\left(T_k(v_n)+h_n\right))\}$ converges weakly in $(L^{p'}(Q))^N$to some F_k, from Proposition 3.4.10, and we obtain that $F_k = A(x,t,\nabla\left(T_k(v)+h\right))$. Hence

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla(T_k(v_n)+h_n)).\nabla(T_k(v_n)+h_n)\\
&\quad\leq \overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla u_n).\nabla T_k(v_n) + \overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla(T_k(v_n)+h_n)).\nabla h_n\\
&\quad\leq \int_Q A(x,t,\nabla(T_k(v)+h)).\nabla(T_k(v)+h).
\end{aligned}$$

As a consequence

$$\{T_k(v_n)\} \text{ converges to } T_k(v), \text{ strongly in } X, \qquad \forall k>0. \tag{3.5.12}$$

Then *to finish the proof we have to check that u is a solution of* (3.1.1). ■

In order to prove (3.5.7) we need a first Lemma, inspired of [32, Lemma 6.1], extending [49, Lemma 6 and Lemma 7] :

Lemma 3.5.3 *Let $\psi_{1,\delta},\psi_{2,\delta}\in C^1(Q)$ be uniformly bounded in $W^{1,\infty}(Q)$ with values in* $[0,1]$, *such that*$\int_Q \psi_{1,\delta}d\mu_s^- \leq \delta$ *and* $\int_Q \psi_{2,\delta}d\mu_s^+ \leq \delta$. *Then,*

$$\frac{1}{m}\int_{\{m\leq v_n<2m\}} |\nabla u_n|^p\psi_{2,\delta} = \omega(n,m,\delta), \qquad \frac{1}{m}\int_{\{m\leq v_n<2m\}} |\nabla v_n|^p\psi_{2,\delta} = \omega(n,m,\delta), \tag{3.5.13}$$

$$\frac{1}{m}\int_{-2m<v_n\leq -m} |\nabla u_n|^p\psi_{1,\delta} = \omega(n,m,\delta), \qquad \frac{1}{m}\int_{-2m<v_n\leq -m} |\nabla v_n|^p\psi_{1,\delta} = \omega(n,m,\delta), \tag{3.5.14}$$

and for any $k>0$,

$$\int_{\{m\leq v_n<m+k\}} |\nabla u_n|^p\psi_{2,\delta} = \omega(n,m,\delta), \qquad \int_{\{m\leq v_n<m+k\}} |\nabla v_n|^p\psi_{2,\delta} = \omega(n,m,\delta), \tag{3.5.15}$$

$$\int\limits_{\{-m-k<v_n\leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n,m,\delta), \qquad \int\limits_{\{-m-k<v_n\leq -m\}} |\nabla v_n|^p \psi_{1,\delta} = \omega(n,m,\delta). \tag{3.5.16}$$

Proof. (i) Proof of (3.5.13), (3.5.14). Set for any $r \in \mathbb{R}$ and any $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left(\frac{-m+\tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m+2h-\tau}{2m+\ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left(\frac{-m+\tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$

Note that $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi, J, S) = (\psi_{2,\delta}, T_1, S_{m,\ell})$ as test functions in (3.4.15) for u_n, and observe that, from (3.5.5),

$$\widehat{\mu_{n,0}} = \mu_{n,0} - (h_n)_t = \widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \tag{3.5.17}$$

Thus we can write $\sum_{i=1}^{6} A_i \leq \sum_{i=7}^{12} A_i$, where

$$A_1 = -\int\limits_{\Omega} \psi_{2,\delta}(0) T_1(S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}), \quad A_2 = -\int_Q (\psi_{2,\delta})_t \overline{T_1}(S_{m,\ell}(v_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) A(x,t,\nabla u_n) \nabla \psi_{2,\delta},$$

$$A_4 = \int_Q (S'_{m,\ell}(v_n))^2 \psi_{2,\delta} T'_1(S_{m,\ell}(v_n)) A(x,t,\nabla u_n) \nabla v_n,$$

$$A_5 = \frac{1}{m} \int\limits_{\{m \leq v_n \leq 2m\}} \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) A(x,t,\nabla u_n) \nabla v_n,$$

$$A_6 = -\frac{1}{2m+\ell} \int\limits_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} A(x,t,\nabla u_n) \nabla v_n,$$

$$A_7 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} f_n, \qquad A_8 = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) g_n . \nabla \psi_{2,\delta},$$

$$A_9 = \int_Q \left(S'_{m,\ell}(v_n)\right)^2 T'_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n . \nabla v_n, \qquad A_{10} = \frac{1}{m} \int\limits_{m \leq v_n \leq 2m} T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} g_n . \nabla v_n,$$

$$A_{11} = -\frac{1}{2m+\ell} \int\limits_{\{2m+\ell \leq v_n < 2(2m+\ell)\}} \psi_{2,\delta} g_n . \nabla v_n, \quad A_{12} = \int_Q S'_{m,\ell}(v_n) T_1(S_{m,\ell}(v_n)) \psi_{2,\delta} d\left(\rho_{n,0} - \eta_{n,0}\right).$$

Since $||S_{m,\ell}(u_{0,n})||_{1,\Omega} \leq \int\limits_{\{m \leq u_{0,n}\}} u_{0,n} dx$, we find $A_1 = \omega(\ell, n, m)$. Otherwise

$$|A_2| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int\limits_{\{m \leq v_n\}} v_n, \qquad |A_3| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int\limits_{\{m \leq v_n\}} \left(|a| + c_2 |\nabla u_n|^{p-1}\right),$$

which implies $A_2 = \omega(\ell, n, m)$ and $A_3 = \omega(\ell, n, m)$. Using (3.4.3) for u_n, we have

$$A_6 = -\int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence $A_6 = \omega(\ell, n, m, \delta)$, since $(\rho_{n,s} - \eta_{n,s})^+$ converges to μ_s^+ as $n \to \infty$ in the narrow topology, and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. We also obtain $A_{11} = \omega(\ell)$ from (3.4.10).

Now $\left\{S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))\right\}_\ell$ converges to $S'_m(v_n)T_1(S_m(v_n))$, $\{S'_m(v_n)T_1(S_m(v_n))\}_n$ converges to $S'_m(v)\ T_1(S_m(v))$, $\{S'_m(v)T_1(S_m(v))\}_m$ converges to 0, $*$-weakly in $L^\infty(Q)$, and $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 3.5.2, we obtain

$$A_7 = \int_Q S'_m(v_n)T_1(S_m(v_n))\psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(v)T_1(S_m(v))\psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m),$$

$$A_8 = \int_Q S'_m(v_n)T_1(S_m(v_n))g_n.\nabla\psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(v)T_1(S_m(v))g\nabla\psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m).$$

Otherwise, $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$, and $\left\{\int_Q \psi_{2,\delta} d\rho_n\right\}$ converges to $\int_Q \psi_{2,\delta} d\mu_s^+$, thus $A_{12} \leq \omega(\ell, n, m, \delta)$.

Using Holder inequality and the condition (3.1.2) we have

$$g_n.\nabla v_n - A(x, t, \nabla u_n)\nabla v_n \leq C_1 \left(|g_n|^{p'} + |\nabla h_n|^p + |a|^{p'}\right)$$

with $C_1 = C_1(p, c_2)$, which implies

$$A_9 - A_4 \leq C_1 \int_Q \left(S'_{m,\ell}(v_n)\right)^2 T'_1(S_{m,\ell}(v_n))\psi_{2,\delta} \left(|g_n|^{p'} + |h_n|^p + |a|^{p'}\right) = \omega(\ell, n, m).$$

Similarly we also show that $A_{10} - A_5/2 \leq \omega(\ell, n, m)$. Combining the estimates, we get $A_5/2 \leq \omega(\ell, n, m, \delta)$. Using Holder inequality we have

$$A(x, t, \nabla u_n)\nabla v_n \geq \frac{c_1}{2}|\nabla u_n|^p - C_2(|a|^{p'} + |\nabla h_n|^p).$$

with $C_2 = C_2(p, c_1, c_2)$, which implies

$$\frac{1}{m} \int\limits_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that for all $m > 4$, $S_{m,\ell}(r) \geq 1$ for any $r \in [\frac{3}{2}m, 2m]$; hence $T_1(S_{m,\ell}(r)) = 1$. So,

$$\frac{1}{m} \int\limits_{\left\{\frac{3}{2}m \leq v_n < 2m\right\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since $|\nabla v_n|^p \leq 2^{p-1}|\nabla u_n|^p + 2^{p-1}|\nabla h_n|^p$, there also holds

$$\frac{1}{m} \int\limits_{\left\{\frac{3}{2}m \leq v_n < 2m\right\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (3.5.13) by summing on each set $\left\{(\frac{4}{3})^i m \leq v_n \leq (\frac{4}{3})^{i+1} m\right\}$ for $i = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$ as test functions in (3.4.15) for u_n, where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (3.5.14).

(ii) Proof of (3.5.15), (3.5.16). We set, for any $k, m, \ell \geq 1$,

$$S_{k,m,\ell}(r) = \int_0^r \left(T_k(\tau - T_m(\tau)) \chi_{[m,k+m+\ell]} + k \frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell))]} \right) d\tau$$

$$S_{k,m}(r) = \int_0^r T_k(\tau - T_m(\tau)) \chi_{[m,\infty)} d\tau.$$

We choose $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$ as test functions in (3.4.15) for u_n. In the same way we also obtain

$$\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that $T_1(S_{k,m,\ell}(r)) = 1$ for any $r \geq m+1$, thus $\int_{\{m+1 \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$, which implies (3.5.15) by changing m into $m-1$. Similarly, we obtain (3.5.16). ■

Next we look at the behaviour near E.

Lemma 3.5.4 *Estimate (3.5.7) holds.*

Proof. There holds

$$I_1 = \int_Q \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla T_k(v_n) - \int_{\{|v_n| \leq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \langle T_k(v) \rangle_\nu .$$

From Proposition 3.4.10, (iv), $\{A(x,t,\nabla (T_k(v_n) + h_n)).\nabla \langle T_k(v) \rangle_\nu\}$ converges weakly in $L^1(Q)$ to $F_k \nabla \langle T_k(v) \rangle_\nu$. And $\left\{\chi_{\{|v_n| \leq k\}}\right\}$ converges to $\chi_{|v| \leq k}$, *a.e.* in Q , and Φ_{δ_1,δ_2} converges to 0 *a.e.* in Q as $\delta_1 \to 0$, and Φ_{δ_1,δ_2} takes its values in $[0,1]$. Thanks to Remark 3.5.2, we have

$$\begin{aligned}
&\int_{\{|v_n| \leq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \langle T_k(v) \rangle_\nu \\
&= \int_Q \chi_{\{|v_n| \leq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla (T_k(v_n) + h_n)).\nabla \langle T_k(v) \rangle_\nu \\
&= \int_Q \chi_{|v| \leq k} \Phi_{\delta_1,\delta_2} F_k.\nabla \langle T_k(v) \rangle_\nu + \omega(n) = \omega(n,\nu,\delta_1).
\end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla T_k(v_n) \leq \omega(n,\delta_1,\delta_2), \tag{3.5.18}$$

then we deduce (3.5.7). As noticed in [32, 49], it is precisely for this estimate that we need the double cut $\psi_{\delta_1}^+\psi_{\delta_2}^+$. To do this, we set, for any $m > k > 0$, and any $r \in \mathbb{R}$,

$$\hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau))\, H_m(\tau) d\tau,$$

where H_m is defined at (3.4.14). Hence supp$\hat{S}_{k,m} \subset [-2m, k]$; and $\hat{S}''_{k,m} = -\chi[-k,k] + \frac{2k}{m}\chi_{[-2m,-m]}$. We choose $(\varphi, S) = (\psi_{\delta_1}^+\psi_{\delta_2}^+, \hat{S}_{k,m})$ as test functions in (3.4.2). From (3.5.17), we can write

$$A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$A_1 = -\int_Q (\psi_{\delta_1}^+\psi_{\delta_2}^+)_t \hat{S}_{k,m}(v_n),\ A_2 = \int_Q (k - T_k(v_n))H_m(v_n)A(x,t,\nabla u_n).\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+),$$

$$A_3 = \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x,t,\nabla u_n).\nabla T_k(v_n),\ A_4 = \frac{2k}{m}\int\limits_{\{-2m<v_n\leq -m\}} \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x,t,\nabla u_n).\nabla v_n,$$

$$A_5 = -\int_Q (k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\widehat{\lambda_{n,0}},\ A_6 = \int_Q (k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\left(\eta_{n,0} - \rho_{n,0}\right);$$

and we estimate A_3. As in [49, p.585], since $\left\{\hat{S}_{k,m}(v_n)\right\}$ converges to $\hat{S}_{k,m}(v)$ weakly in X, and $\hat{S}_{k,m}(v) \in L^\infty(Q)$, and from (3.5.1), there holds

$$A_1 = -\int_Q (\psi_{\delta_1}^+)_t \psi_{\delta_2}^+ \hat{S}_{k,m}(v) - \int_Q \psi_{\delta_1}^+(\psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) + \omega(n) = \omega(n,\delta_1).$$

Next consider A_2. Notice that $v_n = T_{2m}(v_n)$ on supp$(H_m(v_n))$. From Proposition 3.4.10, (iv), the sequence $\left\{A(x,t,\nabla\left(T_{2m}(v_n) + h_n\right)).\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)\right\}$ converges to $F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)$ weakly in $L^1(Q)$. Thanks to Remark 3.5.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ in X to 0 as δ_1 tends to 0, we find

$$A_2 = \int_Q (k - T_k(v))H_m(v)F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+) + \omega(n) = \omega(n,\delta_1).$$

Then consider A_4. Then for some $C = C(p, c_2)$,

$$|A_4| \leq C\frac{2k}{m}\int\limits_{\{-2m<v_n\leq -m\}} \left(|\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'}\right)\psi_{\delta_1}^+\psi_{\delta_2}^+.$$

Since $\psi_{\delta_1}^+$ takes its values in $[0,1]$, from Lemma 3.5.3, we get in particular $A_4 = \omega(n,\delta_1,m,\delta_2)$.

Now estimate A_5. The sequence $\left\{(k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+\right\}$ converges weakly in X to $(k - T_k(v))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+$, and $\{(k - T_k(v_n))H_m(v_n)\}$ converges *-weakly in $L^\infty(Q)$ and *a.e.* in Q to $(k - T_k(v))H_m(v)$. Otherwise $\{f_n\}$ converges to f weakly in $L^1(Q)$ and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. Thanks to Remark 3.5.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ to 0 in X and *a.e.* in Q as $\delta_1 \to 0$, we deduce that

$$A_5 = -\int_Q (k - T_k(v_n))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\widehat{\nu_0} + \omega(n) = \omega(n,\delta_1),$$

where $\widehat{\nu_0} = f - \operatorname{div} g$.

Finally $A_6 \leq 2k \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\eta_n$; using (3.5.2) we also find $A_6 \leq \omega(n, \delta_1, m, \delta_2)$. By addition, since A_3 does not depend on m, we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x,t,\nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Reasoning as before with $(\psi_{\delta_1}^- \psi_{\delta_2}^-, \check{S}_{k,m})$ as test function in (3.4.2), where $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q \psi_{\delta_1}^- \psi_{\delta_2}^- A(x,t,\nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (3.5.18) holds. ■

Next we look at the behaviour far from E.

Lemma 3.5.5 . *Estimate (3.5.8) holds.*

Proof. Here we estimate I_2; we can write

$$I_2 = \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1,\delta_2}) A(x,t,\nabla u_n) \nabla \left(T_k(v_n) - \langle T_k(v)\rangle_\nu\right).$$

Following the ideas of [52], used also in [49], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right).$$

Recall that $\|\langle T_k(v)\rangle_\nu\|_{\infty,Q} \leq k$, and observe that

$$R_{n,\nu,\ell} = 2k \operatorname{sign}(v_n) \quad \text{in } \{|v_n| \geq \ell + 2k\}, \qquad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n,\nu,\ell) \text{ } a.e. \text{ in } Q, \tag{3.5.19}$$

$$\lim_{n\to\infty} R_{n,\nu,\ell} = T_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v - T_k\left(v\right)\right), \qquad a.e. \text{ in } Q, \text{ and weakly in } X. \tag{3.5.20}$$

Next consider $\xi_{1,n_1} \in C_c^\infty([0,T)), \xi_{2,n_2} \in C_c^\infty((0,T])$ with values in $[0,1]$, such that $(\xi_{1,n_1})_t \leq 0$ and $(\xi_{2,n_2})_t \geq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{1,n_2}(t)\}$) converges to 1, for any $t \in [0,T)$ (resp. $t \in (0,T]$) ; and moreover, for any $a \in C([0,T]; L^1(\Omega))$, $\left\{\int_Q a(\xi_{1,n_1})_t\right\}$ and $\int_Q a(\xi_{2,n_2})_t$ converge respectively to $-\int_\Omega a(.,T)$ and $\int_\Omega a(.,0)$. We set

$$\varphi = \varphi_{n,n_1,n_2,l_1,l_2,\ell} = \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})[T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1} - \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})[T_{\ell-k}\left(v_n - T_k(v_n)\right)]_{-l_2}.$$

We can see that

$$\varphi - (1 - \Phi_{\delta_1,\delta_2}) R_{n,\nu,\ell} = \omega(l_1, l_2, n_1, n_2) \quad \text{ in norm in } X \text{ and } a.e. \text{ in } Q. \tag{3.5.21}$$

We can choose $(\varphi, S) = (\varphi_{n,n_1,n_2,l_1,l_2,\ell}, \overline{H_m})$ as test functions in (3.4.7) for u_n, where $\overline{H_m}$ is defined at (3.4.14), with $m > \ell + 2k$. We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$A_1 = \int\limits_{\Omega} \varphi(T)\overline{H_m}(v_n(T))dx, \qquad A_2 = -\int\limits_{\Omega} \varphi(0)\overline{H_m}(u_{0,n})dx,$$
$$A_3 = -\int_Q \varphi_t \overline{H_m}(v_n), \qquad A_4 = \int_Q H_m(v_n)A(x,t,\nabla u_n).\nabla\varphi,$$
$$A_5 = \int_Q \varphi H_m'(v_n)A(x,t,\nabla u_n).\nabla v_n, \qquad A_6 = \int_Q H_m(v_n)\varphi d\widehat{\lambda_{n,0}},$$
$$A_7 = \int_Q H_m(v_n)\varphi d\left(\rho_{n,0} - \eta_{n,0}\right).$$

Estimate of A_4. This term allows to study I_2. Indeed, $\{H_m(v_n)\}$ converges to 1, *a.e.* in Q ; thanks to (3.5.21), (3.5.19) (3.5.20), we have

$$\begin{aligned} A_4 &= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} - \int_Q R_{n,\nu,\ell}A(x,t,\nabla u_n).\nabla\Phi_{\delta_1,\delta_2} + \omega(l_1,l_2,n_1,n_2,m) \\ &= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell) \\ &= I_2 + \int\limits_{\{|v_n|>k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell), \end{aligned}$$

where

$$B_1 = \int\limits_{\{|v_n|>k\}} (1-\Phi_{\delta,\eta})(\chi_{\left|v_n-\langle T_k(v)\rangle_\nu\right|\le \ell+k} - \chi_{||v_n|-k|\le \ell-k})A(x,t,\nabla u_n).\nabla v_n,$$
$$B_2 = -\int\limits_{\{|v_n|>k\}} (1-\Phi_{\delta_1,\delta_2})\chi_{\left|v_n-\langle T_k(v)\rangle_\nu\right|\le \ell+k}A(x,t,\nabla u_n).\nabla\langle T_k(v)\rangle_\nu.$$

Now $\{A(x,t,\nabla\left(T_{\ell+2k}(v_n)+h_n\right)).\nabla\langle T_k(v)\rangle_\nu\}$ converges to $F_{\ell+2k}\nabla\langle T_k(v)\rangle_\nu$, weakly in $L^1(Q)$. Otherwise $\left\{\chi_{|v_n|>k}\chi_{\left|v_n-\langle T_k(v)\rangle_\nu\right|\le \ell+k}\right\}$ converges to $\chi_{|v|>k}\chi_{\left|v-\langle T_k(v)\rangle_\nu\right|\le \ell+k}$, *a.e.* in Q. And $\{\langle T_k(v)\rangle_\nu\}$ converges to $T_k(v)$ strongly in X. Thanks to Remark 3.5.2 we get

$$\begin{aligned} B_2 &= -\int_Q \left(1-\Phi_{\delta_1,\delta_2}\right)\chi_{|v|>k}\,\chi_{\left|v-\langle T_k(v)\rangle_\nu\right|\le \ell+k}F_{\ell+2k}.\nabla\langle T_k(v)\rangle_\nu + \omega(n) \\ &= -\int_Q \left(1-\Phi_{\delta_1,\delta_2}\right)\chi_{|v|>k}\,\chi_{|v-T_k(v)|\le \ell+k}F_{\ell+2k}.\nabla T_k(v) + \omega(n,\nu) = \omega(n,\nu), \end{aligned}$$

since $\nabla T_k(v)\,\chi_{|v|>k} = 0$. Besides, we see that, for some $C = C(p,c_2)$,

$$|B_1| \le C \int\limits_{\{\ell-2k\le|v_n|<\ell+2k\}} \left(1-\Phi_{\delta_1,\delta_2}\right)\left(|\nabla u_n|^p + |\nabla v_n|^p + |a|^{p'}\right).$$

Using (3.5.3) and (3.5.4) and applying (3.5.15) and (3.5.16) to $1-\Phi_{\delta_1,\delta_2}$, we obtain, for $k>0$

$$\int_{\{m\leq|v_n|<m+4k\}} (|\nabla u_n|^p+|\nabla v_n|^p)(1-\Phi_{\delta_1,\delta_2})=\omega(n,m,\delta_1,\delta_2). \tag{3.5.22}$$

Thus, $B_1=\omega(n,\nu,\ell,\delta_1,\delta_2)$, hence $B_1+B_2=\omega(n,\nu,\ell,\delta_1,\delta_2)$. Then

$$A_4=I_2+\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2). \tag{3.5.23}$$

Estimate of A_5. For $m>\ell+2k$, since $|\varphi|\leq 2\ell$, and (3.5.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned}A_5&=\int_Q(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}H_m'(v_n)A(x,t,\nabla u_n).\nabla v_n+\omega(l_1,l_2,n_1,n_2)\\&=-\frac{2k}{m}\int_{\{m\leq|v_n|<2m\}}(1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla v_n+\omega(l_1,l_2,n_1,n_2);\end{aligned}$$

here, the final equality followed from the relation, since $m>\ell+2k$,

$$R_{n,\nu,\ell}H_m'(v_n)=-\frac{2k}{m}\chi_{m\leq|v_n|\leq 2m},\quad a.e. \text{ in } Q. \tag{3.5.24}$$

Next we go to the limit in m, by using (3.4.3), (3.4.4) for u_n, with $\phi=(1-\Phi_{\delta_1,\delta_2})$. There holds

$$A_5=-2k\int_Q(1-\Phi_{\delta_1,\delta_2})d\left((\rho_{n,s}-\eta_{n,s})^++(\rho_{n,s}-\eta_{n,s})^-\right)+\omega(l_1,l_2,n_1,n_2,m).$$

Then, from (3.5.3) and (3.5.4), we get $A_5=\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of A_6. Again, from (3.5.21),

$$\begin{aligned}A_6&=\int_Q H_m(v_n)\varphi f_n+\int_Q g_n.\nabla(H_m(v_n)\varphi)\\&=\int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}f_n+\int_Q g_n.\nabla(H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell})+\omega(l_1,l_2,n_1,n_2).\end{aligned}$$

Thus we can write $A_6=D_1+D_2+D_3+D_4+\omega(l_1,l_2,n_1,n_2)$, where

$$D_1=\int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}f_n,\qquad D_2=\int_Q(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}H_m'(v_n)g_n.\nabla v_n,$$

$$D_3=\int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})g_n.\nabla R_{n,\nu,\ell},\qquad D_4=-\int_Q H_m(v_n)R_{n,\nu,\ell}g_n.\nabla\Phi_{\delta_1,\delta_2}.$$

Since $\{f_n\}$ converges to f weakly in $L^1(Q)$, and (3.5.19)-(3.5.20) hold, we get from Remark 3.5.2,

$$D_1=\int_Q(1-\Phi_{\delta_1,\delta_2})\left(T_{\ell+k}\left(v-\langle T_k(v)\rangle_\nu\right)-T_{\ell-k}\left(v-T_k\left(v\right)\right)\right)f+\omega(m,n)=\omega(m,n,\nu,\ell).$$

We deduce from (3.4.10) that $D_2 = \omega(m)$. Next consider D_3. Note that $H_m(v_n) = 1 + \omega(m)$, and (3.5.20) holds, and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, and $\langle T_k(v)\rangle_\nu$ converges to $T_k(v)$ strongly in X. Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1-\Phi_{\delta_1,\delta_2}) g.\nabla \left(T_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v - T_k\left(v\right)\right)\right) + \omega(m,n) \\ &= \int_Q (1-\Phi_{\delta_1,\delta_2}) g.\nabla \left(T_{\ell+k}\left(v - T_k(v)\right) - T_{\ell-k}\left(v - T_k\left(v\right)\right)\right) + \omega(m,n,\nu) \\ &= \omega(m,n,\nu,\ell). \end{aligned}$$

Similarly we also get $D_4 = \omega(m,n,\nu,\ell)$. Thus $A_6 = \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of A_7. We have

$$\begin{aligned} |A_7| &= \left| \int_Q S_m'(v_n)\left(1-\Phi_{\delta_1,\delta_2}\right) R_{n,\nu,\ell} d\left(\rho_{n,0} - \eta_{n,0}\right)\right| + \omega(l_1,l_2,n_1,n_2) \\ &\le 4k \int_Q \left(1-\Phi_{\delta_1,\delta_2}\right) d\left(\rho_n + \eta_n\right) + \omega(l_1,l_2,n_1,n_2). \end{aligned}$$

From (3.5.3) and (3.5.4) we get $A_7 = \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of $A_1 + A_2 + A_3$. We set

$$J(r) = T_{\ell-k}\left(r - T_k\left(r\right)\right), \qquad \forall r \in \mathbb{R},$$

and use the notations $\overline{J}$ and $\mathcal{J}$ of (3.4.11). From the definitions of ξ_{1,n_1}, ξ_{1,n_2}, we can see that

$$\begin{aligned} A_1 + A_2 &= -\int_\Omega J(v_n(T))\overline{H_m}(v_n(T)) - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)\overline{H_m}(u_{0,n}) + \omega(l_1,l_2,n_1,n_2) \\ &= -\int_\Omega J(v_n(T))v_n(T) - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)u_{0,n} + \omega(l_1,l_2,n_1,n_2,m), \end{aligned} \tag{3.5.25}$$

where $z_\nu = \langle T_k(v)\rangle_\nu(0)$. We can write $A_3 = F_1 + F_2$, where

$$\begin{aligned} F_1 &= -\int_Q \left(\xi_{n_1}(1-\Phi_{\delta_1,\delta_2})[T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1}\right)_t \overline{H_m}(v_n), \\ F_2 &= \int_Q \left(\xi_{n_2}(1-\Phi_{\delta_1,\delta_2})[T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right)]_{-l_2}\right)_t \overline{H_m}(v_n). \end{aligned}$$

Estimate of F_2. We write $F_2 = G_1 + G_2 + G_3$, with

$$\begin{aligned} G_1 &= -\int_Q (\Phi_{\delta_1,\delta_2})_t \xi_{n_2}[T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right)]_{-l_2}\overline{H_m}(v_n), \\ G_2 &= \int_Q (1-\Phi_{\delta_1,\delta_2})(\xi_{n_2})_t[T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right)]_{-l_2}\overline{H_m}(v_n), \\ G_3 &= \int_Q \xi_{n_2}(1-\Phi_{\delta_1,\delta_2})\left([T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right)]_{-l_2}\right)_t \overline{H_m}(v_n). \end{aligned}$$

We find easily

$$G_1 = -\int_Q (\Phi_{\delta_1,\delta_2})_t J(v_n)v_n + \omega(l_1,l_2,n_1,n_2,m),$$

$$G_2 = \int_Q (1-\Phi_{\delta_1,\delta_2})(\xi_{n_2})_t J(v_n)\overline{H_m}(v_n) + \omega(l_1,l_2) = \int_\Omega J(u_{0,n})u_{0,n} + \omega(l_1,l_2,n_1,n_2,m).$$

Next consider G_3. Setting $b = \overline{H_m}(v_n)$, there holds from (3.4.13) and (3.4.12),

$$(([J(b)]_{-l_2})_t b)(.,t) = \frac{b(.,t)}{l_2}(J(b)(.,t) - J(b)(.,t-l_2)).$$

Hence

$$\left([T_{\ell-k}(v_n - T_k(v_n))]_{-l_2}\right)_t \overline{H_m}(v_n) \geq \left(\left[\mathcal{J}(\overline{H_m}(v_n))\right]_{-l_2}\right)_t = \left([\mathcal{J}(v_n)]_{-l_2}\right)_t,$$

since $\mathcal{J}$ is constant in $\{|r| \geq m+\ell+2k\}$. Integrating by parts in G_3, we find

$$\begin{aligned}
G_3 &\geq \int_Q \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})\left([\mathcal{J}(v_n)]_{-l_2}\right)_t \\
&= -\int_Q (\xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2}))_t[\mathcal{J}(v_n)]_{-l_2} + \int_\Omega \xi_{2,n_2}(T)[\mathcal{J}(v_n)]_{-l_2}(T)dx \\
&= -\int_Q (\xi_{2,n_2})_t(1-\Phi_{\delta_1,\delta_2})\mathcal{J}(v_n) \\
&\quad + \int_Q \xi_{2,n_2}(\Phi_{\delta_1,\delta_2})_t\mathcal{J}(v_n) + \int_\Omega \xi_{2,n_2}(T)\mathcal{J}(v_n(T)) + \omega(l_1,l_2) \\
&= -\int_\Omega \mathcal{J}(u_{0,n})dx + \int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{J}(v_n) + \int_\Omega \mathcal{J}(v_n(T)) + \omega(l_1,l_2,n_1,n_2).
\end{aligned}$$

Therefore, since $\mathcal{J}(v_n) - J(v_n)v_n = -\overline{J}(v_n)$ and $\overline{J}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$, we obtain

$$F_2 \geq \int_\Omega \overline{J}(u_{0,n})dx - \int_Q (\Phi_{\delta_1,\delta_2})_t\overline{J}(v_n) + \int_\Omega \mathcal{J}(v_n(T))dx + \omega(l_1,l_2,n_1,n_2,m). \qquad (3.5.26)$$

Estimate of F_1. Since $m > \ell+2k$, there holds $T_{\ell+k}(v_n - \langle T_k(v)\rangle_\nu) = T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)$ on supp$\overline{H_m}(v_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$L_1 = -\int_Q \left(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\left[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\right]_{l_1}\right)_t \left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)\rangle_\nu\right)$$

$$L_2 = -\int_Q \left(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\left[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\right]_{l_1}\right)_t \langle T_k(\overline{H_m}(v))\rangle_\nu.$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$\begin{aligned}
L_2 = &\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\left(\langle T_k(\overline{H_m}(v))\rangle_\nu\right)_t \\
&+\int_\Omega \xi_{1,n_1}(0)\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}(0)\langle T_k(\overline{H_m}(v))\rangle_\nu(0) \\
&=\nu\int_Q (1-\Phi_{\delta_1,\delta_2})T_{\ell+k}\left(v_n-\langle T_k(v)\rangle_\nu\right)\left(T_k(v)-\langle T_k(v)\rangle_\nu\right) \\
&+\int_\Omega T_{\ell+k}\left(u_{0,n}-z_\nu\right)z_\nu dx+\omega(l_1,l_2,n_1,n_2).
\end{aligned} \tag{3.5.27}$$

We decompose L_1 into $L_1 = K_1 + K_2 + K_3$, where

$$\begin{aligned}
K_1 &= -\int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right) \\
K_2 &= \int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right) \\
K_3 &= -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\Big(\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\Big)_t\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v)\rangle_\nu\right).
\end{aligned}$$

Then we check easily that

$$K_1 = \int_\Omega T_{\ell+k}\left(v_n-\langle T_k(v)\rangle_\nu\right)(T)\left(v_n-\langle T_k(v)\rangle_\nu\right)(T)dx+\omega(l_1,l_2,n_1,n_2,m),$$

$$K_2 = \int_Q (\Phi_{\delta_1,\delta_2})_t T_{\ell+k}\left(v_n-\langle T_k(v)\rangle_\nu\right)\left(v_n-\langle T_k(v)\rangle_\nu\right)+\omega(l_1,l_2,n_1,n_2,m).$$

Next consider K_3. Here we use the function $\mathcal{T}_k$ defined at (3.4.13).
We set $b = \overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu$. Hence from (3.4.12),

$$\begin{aligned}
(([T_{\ell+k}(b)]_{l_1})_t b)(.,t) &= \frac{b(.,t)}{l_1}(T_{\ell+k}(b)(.,t+l_1)-T_{\ell+k}(b)(.,t)) \\
&\le \frac{1}{l_1}(\mathcal{T}_{\ell+k}(b)((.,t+l_1))-\mathcal{T}_{\ell+k}(b)(.,t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big(\big[T_{\ell+k}\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\Big)_t\left(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu\right) \\
&\quad\le \Big(\big[\mathcal{T}_{\ell+k}(\overline{H_m}(v_n)-\langle T_k(\overline{H_m}(v))\rangle_\nu)\big]_{l_1}\Big)_t = \Big([\mathcal{T}_{\ell+k}(v_n-\langle T_k(v)\rangle_\nu]_{l_1}\Big)_t.
\end{aligned}$$

Then

$$\begin{aligned}K_3 &\geq -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\Big([\mathcal{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)]_{l_1}\Big)_t \\ &= \int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})[\mathcal{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)]_{l_1} - \int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t[\mathcal{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)]_{l_1} \\ &+ \int_\Omega \xi_{1,n_1}(0)[\mathcal{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)]_{l_1}(0)dx \\ &= -\int_\Omega \mathcal{T}_{\ell+k}\,(v_n(T) - \langle T_k(v)\rangle_\nu(T))\,dx - \int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu) \\ &+ \int_\Omega \mathcal{T}_{\ell+k}\,(u_{0,n} - z_\nu)\,dx + \omega(l_1,l_2,n_1,n_2).\end{aligned}$$

We find by addition, since $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \overline{T}_{\ell+k}(r)$ for any $r \in \mathbb{R}$,

$$\begin{aligned}L_1 \geq &\int_\Omega \mathcal{T}_{\ell+k}\,(u_{0,n} - z_\nu)\,dx + \int_\Omega \overline{T}_{\ell+k}\,(v_n(T) - \langle T_k(v)\rangle_\nu(T))\,dx \\ &+ \int_Q (\Phi_{\delta_1,\delta_2})_t\overline{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu) + \omega(l_1,l_2,n_1,n_2,m).\end{aligned} \tag{3.5.28}$$

We deduce from (3.5.28), (3.5.27), (3.5.26),

$$\begin{aligned}A_3 \geq &\int_\Omega \overline{J}(u_{0,n}) + \int_\Omega \mathcal{T}_{\ell+k}\,(u_{0,n} - z_\nu)\,dx + \int_\Omega T_{\ell+k}\,(u_{0,n} - z_\nu)\,z_\nu dx \\ &+ \int_\Omega \overline{T}_{\ell+k}\,(v_n(T) - \langle T_k(v)\rangle_\nu(T)) + \int_\Omega \mathcal{J}(v_n(T))dx + \int_Q (\Phi_{\delta_1,\delta_2})_t\left(\overline{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu) - \overline{J}(v_n)\right) \\ &+ \nu\int_Q (1-\Phi_{\delta_1,\delta_2})T_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)\,(T_k(v) - \langle T_k(v)\rangle_\nu) + \omega(l_1,l_2,n_1,n_2,m).\end{aligned} \tag{3.5.29}$$

Next we add (3.5.25) and (3.5.29). Note that $\mathcal{J}(v_n(T)) - J(v_n(T))v_n(T) = -\overline{J}(v_n(T))$, and also $\mathcal{T}_{\ell+k}\,(u_{0,n} - z_\nu) - T_{\ell+k}\,(u_{0,n} - z_\nu)\,(z_\nu - u_{0,n}) = -\overline{T}_{\ell+k}\,(u_{0,n} - z_\nu)$. Then we find

$$\begin{aligned}A_1 + A_2 + A_3 \geq &\int_\Omega \left(\overline{J}(u_{0,n}) - \overline{T}_{\ell+k}\,(u_{0,n} - z_\nu)\right)dx + \int_\Omega \left(\overline{T}_{\ell+k}\,(v_n(T) - \langle T_k(v)\rangle_\nu(T)) - \overline{J}(v_n(T))\right)dx \\ &+ \int_Q (\Phi_{\delta_1,\delta_2})_t\left(\overline{T}_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu) - \overline{J}(v_n)\right) \\ &+ \nu\int_Q (1-\Phi_{\delta_1,\delta_2})T_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)\,(T_k(v) - \langle T_k(v)\rangle_\nu) + \omega(l_1,l_2,n_1,n_2,m).\end{aligned}$$

Notice that $\overline{T}_{\ell+k}\,(r-s) - \overline{J}(r) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$; thus

$$\int_\Omega \left(\overline{T}_{\ell+k}\,(v_n(T) - \langle T_k(v)\rangle_\nu(T)) - \overline{J}(v_n(T))\right)dx \geq 0.$$

And $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$ and $\{v_n\}$ converges to v in $L^1(Q)$ from Proposition 3.4.10. Thus we obtain

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v)\right)$$

$$+\nu \int_Q \left(1 - \Phi_{\delta_1,\delta_2}\right) T_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right)\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m, n).$$

Moreover $T_{\ell+k}(r-s)(T_k(r) - s) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$, hence

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v)\right)$$
$$+\omega(l_1, l_2, n_1, n_2, m, n).$$

As $\nu \to \infty$, $\{z_\nu\}$ converges to $T_k(u_0)$, *a.e.* in Ω, thus we get

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - T_k(u_0)\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(v - T_k(v)\right) - \overline{J}(v)\right)$$
$$+ \omega(l_1, l_2, n_1, n_2, m, n, \nu).$$

Finally $\left|\overline{T}_{\ell+k}(r - T_k(r)) - \overline{J}(r)\right| \leq 2k|r|\chi_{\{|r|\geq\ell\}}$ for any $r \in \mathbb{R}$, thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ which implies (3.5.8), since I_2 does not depend on $l_1, l_2, n_1, n_2, m, \ell$. ■

Next we conclude the proof of Theorem 3.2.1 :

Lemma 3.5.6 *The function u is a R-solution of (3.1.1).*

Proof. (i) First show that u satisfies (3.4.2). Here we proceed as in [49]. Let $\varphi \in X \cap L^\infty(Q)$ such $\varphi_t \in X' + L^1(Q)$, $\varphi(.,T) = 0$, and $S \in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on $\mathbb{R}$, $S(0) = 0$. Let $M > 0$ such that supp$S' \subset [-M, M]$. Taking successively (φ, S) and $(\varphi\psi_\delta^\pm, S)$ as test functions in (3.4.2) applied to u_n, we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \qquad A_{2,\delta,\pm} + A_{3,\delta,\pm} + A_{4,\delta,\pm} = A_{5,\delta,\pm} + A_{6,\delta,\pm} + A_{7,\delta,\pm},$$

where

$$A_1 = -\int_\Omega \varphi(0) S(u_{0,n}), \quad A_2 = -\int_Q \varphi_t S(v_n), \quad A_{2,\delta,\pm} = -\int_Q (\varphi\psi_\delta^\pm)_t S(v_n),$$

$$A_3 = \int_Q S'(v_n) A(x,t,\nabla u_n).\nabla\varphi, \quad A_{3,\delta,\pm} = \int_Q S'(v_n) A(x,t,\nabla u_n).\nabla(\varphi\psi_\delta^\pm),$$

$$A_4 = \int_Q S''(v_n)\varphi A(x,t,\nabla u_n).\nabla v_n, \quad A_{4,\delta,\pm} = \int_Q S''(v_n)\varphi\psi_\delta^\pm A(x,t,\nabla u_n).\nabla v_n,$$

$$A_5 = \int_Q S'(v_n)\varphi d\widehat{\lambda_{n,0}}, \quad A_6 = \int_Q S'(v_n)\varphi d\rho_{n,0}, \quad A_7 = -\int_Q S'(v_n)\varphi d\eta_{n,0},$$

$$A_{5,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}}, \quad A_{6,\delta,\pm} = \int_Q S'(v_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7,\delta,\pm} = -\int_Q S'(v_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$, and $\{S(v_n)\}$ converges to $S(v)$ strongly in X and weak* in $L^\infty(Q)$, there holds, from (3.5.2),

$$A_1 = -\int_\Omega \varphi(0)S(u_0) + \omega(n), \quad A_2 = -\int_Q \varphi_t S(v) + \omega(n), \quad A_{2,\delta,\psi_\delta^\pm} = \omega(n,\delta).$$

Moreover $T_M(v_n)$ converges to $T_M(v)$, then $T_M(v_n) + h_n$ converges to $T_k(v) + h$ strongly in X, thus

$$\begin{aligned} A_3 &= \int_Q S'(v_n)A(x,t,\nabla\left(T_M\left(v_n\right) + h_n\right)).\nabla\varphi \\ &= \int_Q S'(v)A(x,t,\nabla\left(T_M\left(v\right) + h\right)).\nabla\varphi + \omega(n) \\ &= \int_Q S'(v)A(x,t,\nabla u).\nabla\varphi + \omega(n); \end{aligned}$$

and

$$\begin{aligned} A_4 &= \int_Q S''(v_n)\varphi A(x,t,\nabla\left(T_M\left(v_n\right) + h_n\right)).\nabla T_M\left(v_n\right) \\ &= \int_Q S''(v)\varphi A(x,t,\nabla\left(T_M\left(v\right) + h\right)).\nabla T_M\left(v\right) + \omega(n) \\ &= \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v + \omega(n). \end{aligned}$$

In the same way, since $\psi_\delta^\pm$ converges to 0 in X,

$$\begin{aligned} A_{3,\delta,\pm} &= \int_Q S'(v)A(x,t,\nabla u).\nabla(\varphi\psi_\delta^\pm) + \omega(n) = \omega(n,\delta), \\ A_{4,\delta,\pm} &= \int_Q S''(v)\varphi\psi_\delta^\pm A(x,t,\nabla u).\nabla v + \omega(n) = \omega(n,\delta). \end{aligned}$$

And $\{g_n\}$ converges strongly in $(L^{p'}(\Omega))^N$, thus

$$\begin{aligned} A_5 &= \int_Q S'(v_n)\varphi f_n + \int_Q S'(v_n)g_n.\nabla\varphi + \int_Q S''(v_n)\varphi g_n.\nabla T_M(v_n) \\ &= \int_Q S'(v)\varphi f + \int_Q S'(v)g.\nabla\varphi + \int_Q S''(v)\varphi g.\nabla T_M(v) + \omega(n) \\ &= \int_Q S'(v)\varphi d\widehat{\mu_0} + \omega(n). \end{aligned}$$

and $A_{5,\delta,\pm} = \int_Q S'(v)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}} + \omega(n) = \omega(n,\delta)$. Then $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n,\delta)$. From (3.5.2) we verify that $A_{7,\delta,+} = \omega(n,\delta)$ and $A_{6,\delta,-} = \omega(n,\delta)$. Moreover, from (3.5.6) and (3.5.2), we find

$$|A_6 - A_{6,\delta,+}| \leq \int_Q \left|S'(v_n)\varphi\right|(1-\psi_\delta^+)d\rho_{n,0} \leq \|S\|_{W^{2,\infty}(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\int_Q (1-\psi_\delta^+)d\rho_n = \omega(n,\delta).$$

Similarly we also have $|A_7 - A_{7,\delta,-}| \leq \omega(n,\delta)$. Hence $A_6 = \omega(n)$ and $A_7 = \omega(n)$. Therefore, we finally obtain (3.4.2) :

$$-\int_{\Omega} \varphi(0)S(u_0)dx - \int_Q \varphi_t S(v)$$
$$+\int_Q S'(v)A(x,t,\nabla u).\nabla\varphi + \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v = \int_Q S'(v)\varphi d\widehat{\mu_0}. \quad (3.5.30)$$

(ii) Next, we prove (3.4.3) and (3.4.4). We take $\varphi \in C_c^\infty(Q)$ and take $((1-\psi_\delta^-)\varphi, \overline{H_m})$ as test functions in (3.5.30), with $\overline{H_m}$ as in (3.4.14). We can write $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$, where

$$\begin{aligned}
D_{1,m} &= -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t \overline{H_m}(v), \\
D_{2,m} &= \int_Q H_m(v)A(x,t,\nabla u).\nabla\left((1-\psi_\delta^-)\varphi\right), \\
D_{3,m} &= \int_Q H_m(v)(1-\psi_\delta^-)\varphi d\widehat{\mu_0}, \\
D_{4,m} &= \frac{1}{m}\int_{m\leq v\leq 2m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u).\nabla v, \\
D_{5,m} &= -\frac{1}{m}\int_{-2m\leq v\leq -m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u)\nabla v.
\end{aligned} \quad (3.5.31)$$

Taking the same test functions in (3.4.2) applied to u_n, there holds $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$, where

$$\begin{aligned}
D_{1,m}^n &= -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t \overline{H_m}(v_n), \\
D_{2,m}^n &= \int_Q H_m(v_n)A(x,t,\nabla u_n).\nabla\left((1-\psi_\delta^-)\varphi\right), \\
D_{3,m}^n &= \int_Q H_m(v_n)(1-\psi_\delta^-)\varphi d(\widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0}), \\
D_{4,m}^n &= \frac{1}{m}\int_{m\leq v\leq 2m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u_n).\nabla v_n, \\
D_{5,m}^n &= -\frac{1}{m}\int_{-2m\leq v_n\leq -m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u_n).\nabla v_n.
\end{aligned} \quad (3.5.32)$$

In (3.5.32), we go to the limit as $m \to \infty$. Since $\{\overline{H}_m(v_n)\}$ converges to v_n and $\{H_m(v_n)\}$ converges to 1, *a.e.* in Q, and $\{\nabla H_m(v_n)\}$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation $D_1^n + D_2^n = D_3^n + D^n$, where

$$D_1^n = -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t v_n, \ D_2^n = \int_Q A(x,t,\nabla u_n)\nabla\left((1-\psi_\delta^-)\varphi\right), \ D_3^n = \int_Q (1-\psi_\delta^-)\varphi d\widehat{\lambda_{n,0}}$$

$$\begin{aligned}
D^n &= \int_Q (1-\psi_\delta^-)\varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1-\psi_\delta^-)\varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\
&= \int_Q (1-\psi_\delta^-)\varphi d(\rho_n - \eta_n).
\end{aligned}$$

Clearly, $D_{i,m} - D_i^n = \omega(n,m)$ for $i = 1,2,3$. From Lemma (3.5.3) and (3.5.2)-(3.5.4), we obtain $D_{5,m} = \omega(n,m,\delta)$, and

$$\frac{1}{m} \int\limits_{\{m \leq v < 2m\}} \psi_\delta^- \varphi A(x,t,\nabla u).\nabla v = \omega(n,m,\delta),$$

thus,

$$D_{4,m} = \frac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi A(x,t,\nabla u).\nabla v + \omega(n,m,\delta).$$

Since $\left| \int_Q (1-\psi_\delta^-)\varphi d\eta_n \right| \leq \|\varphi\|_{L^\infty} \int_Q (1-\psi_\delta^-) d\eta_n$, it follows that $\int_Q (1-\psi_\delta^-)\varphi d\eta_n = \omega(n,m,\delta)$ from (3.5.4). And $\left| \int_Q \psi_\delta^- \varphi d\rho_n \right| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$, thus, from (3.5.2),

$$\int_Q (1-\psi_\delta^-)\varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta).$$

Then $D^n = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta)$. Therefore by substraction, we get

$$\frac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta),$$

hence

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+, \tag{3.5.33}$$

which proves (3.4.3) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\lim_{m\to\infty} \tfrac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi A(x,t,\nabla u).\nabla v$$

$$= \lim_{m\to\infty} \tfrac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi \psi_\delta^+ A(x,t,\nabla u)\nabla v + \lim_{m\to\infty} \tfrac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi(1-\psi_\delta^+) A(x,t,\nabla u).\nabla v$$

$$= \int_Q \varphi \psi_\delta^+ d\mu_s^+ + \lim_{m\to\infty} \tfrac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi(1-\psi_\delta^+) A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+ + D,$$

where,

$$D = \int_Q \varphi(1-\psi_\delta^+) d\mu_s^+ + \lim_{n\to\infty} \frac{1}{m} \int\limits_{\{m \leq v < 2m\}} \varphi(1-\psi_\delta^+) A(x,t,\nabla u).\nabla v = \omega(\delta).$$

Therefore, (3.5.33) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (3.4.3) by density, and similarly, (3.4.4). This completes the proof of Theorem 3.2.1. ∎

As a consequence of Theorem 3.2.1, we get the following :

Corollary 3.5.7 *Let $u_0 \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_b(Q)$. Then there exists a R-solution u to the problem 3.1.1 with data (μ, u_0). Furthermore, if $v_0 \in L^1(\Omega)$ and $\omega \in \mathfrak{M}_b(Q)$ such that $u_0 \leq v_0$ and $\mu \leq \omega$, then one can find R-solution v to the problem 3.1.1 with data (ω, v_0) such that $u \leq v$.*

In particular, if $a \equiv 0$ in (3.1.2), then u satisfies (3.4.21) and $\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq M$ with $M = ||u_0||_{1,\Omega} + |\mu|(Q)$.

3.6 Equations with perturbation terms

Let A be a Caratheodory function on $Q \times \mathbb{R}^N$ and satisfy (3.1.2), (3.1.3) with $a \equiv 0$. Let $\mathcal{G} : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function. If U is a function defined in Q we define the function $\mathcal{G}(U)$ in Q by

$$\mathcal{G}(U)(x,t) = \mathcal{G}(x,t,U(x,t)) \qquad \text{for } a.e.\ (x,t) \in Q.$$

We consider the problem (3.1.5) :

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 \qquad \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 \qquad \text{in } \Omega, \end{cases}$$

where $\mu \in \mathfrak{M}_b(Q)$, $u_0 \in L^1(\Omega)$. We say that u is a R-solution of problem (3.1.5) if $\mathcal{G}(u) \in L^1(Q)$ and u is a R-solution of (3.1.1) with data $(\mu - \mathcal{G}(u), u_0)$.

3.6.1 Subcritical type results

For proving Theorem 3.2.2, we begin by an integration Lemma :

Lemma 3.6.1 *Let G satisfying (3.2.3). If a measurable function V in Q satisfies*

$$\text{meas}\,\{|V| \geq t\} \leq M t^{-p_c}, \qquad \forall t \geq 1,$$

for some $M > 0$, then for any $L > 1$,

$$\int\limits_{\{|V| \geq L\}} G(|V|) \leq p_c M \int_L^\infty G(s)\, s^{-1-p_c} ds. \tag{3.6.1}$$

Proof. Indeed, setting $G_L(s) = \chi_{[L,\infty)}(s) G(s)$, we have

$$\int\limits_{\{|V| \geq L\}} G(|V|) dxdt = \int_Q G_L(|V|) dxdt \leq \int_0^\infty G_L(|V|^*(s)) ds$$

where $|V|^*$ is and the rearrangement of $|V|$, defined by

$$|V|^*(s) = \inf\{a > 0 : \text{meas}\,\{|V| > a\}) \leq s\}, \qquad \forall s \geq 0.$$

From the assumption, we get $|V|^*(s) \leq \sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)$. Thus, for any $L > 1$,

$$\int\limits_{\{|V|\geq L\}} G(|V|)dxdt \leq \int_0^\infty G_L\left(\sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)\right) ds = p_c M \int_L^\infty G(s)\, s^{-1-p_c} ds,$$

which implies (3.6.1). ■

Proof of Theorem 3.2.2. Proof of (i) Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(Q)$, with $\mu_0 \in \mathfrak{M}_0(Q), \mu_s \in \mathfrak{M}_s(Q)$, and $u_0 \in L^1(\Omega)$. By Proposition 3.3.1, we can find $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$ which strongly converge to f_i, g_i, h_i in $L^1(Q), (L^{p'}(Q))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, for $i = 1, 2$, such that $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$, and $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$, converging respectively for $i = 1, 2$ to μ_0^+, μ_0^- in the narrow topology; and we can find nonnegative $\mu_{n,s,i} \in C_c^\infty(Q), i = 1, 2$, converging respectively to μ_s^+, μ_s^- in the narrow topology.

Furthermore, if we set

$$\mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2},$$

then $|\mu_n|(Q) \leqq |\mu|(Q)$. Consider a sequence $\{u_{0,n}\} \subset C_c^\infty(\Omega)$ which strongly converges to u_0 in $L^1(\Omega)$ and satisfies $||u_{0,n}||_{1,\Omega} \leqq ||u_0||_{1,\Omega}$.

Let u_n be a solution of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) + \mathcal{G}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} \quad \text{in } \Omega. \end{cases}$$

We can choose $\varphi = \varepsilon^{-1} T_\varepsilon(u_n)$ as test function of above problem. Then we find

$$\int_Q \left(\varepsilon^{-1}\overline{T_\varepsilon}(u_n)\right)_t + \int_Q \varepsilon^{-1} A(x,t,\nabla T_\varepsilon(u_n)).\nabla T_\varepsilon(u_n) + \int_Q \mathcal{G}(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) = \int_Q \varepsilon^{-1}T_\varepsilon(u_n)d\mu_n.$$

Since

$$\int_Q \left(\varepsilon^{-1}\overline{T_\varepsilon}(u_n)\right)_t = \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_n(T))dx - \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_{0,n})dx \geq -||u_{0,n}||_{1,\Omega},$$

there holds

$$\int_Q \mathcal{G}(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) \leq |\mu_n|(Q) + ||u_{0,n}||_{L^1(\Omega)} \leq |\mu|(Q) + ||u_0||_{1,\Omega}.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_Q |\mathcal{G}(x,t,u_n)| \leq |\mu|(Q) + ||u_0||_{1,\Omega}. \tag{3.6.2}$$

Next apply Proposition 3.4.8 and Remark 3.4.9 to u_n with initial data $u_{0,n}$ and measure data $\mu_n - \mathcal{G}(u_n) \in L^1(Q)$, we get

$$\text{meas}\,\{|u_n| \geq s\} \leq C(|\mu|(Q) + ||u_0||_{L^1(\Omega)})^{\frac{p+N}{N}} s^{-p_c}, \qquad \forall s > 0, \forall n \in \mathbb{N},$$

for some $C = C(N, p, c_1, c_2)$. Since $|\mathcal{G}(x,t,u_n)| \leq G(|u_n|)$, we deduce from (3.6.1) that $\{|\mathcal{G}(u_n)|\}$ is equi-integrable. Then, thanks to Proposition 3.4.10, up to a subsequence, $\{u_n\}$ converges to some function u, *a.e.* in Q, and $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$. Therefore, by Theorem 3.2.1, u is a R-solution of (3.2.4).

Proof of (ii). Let $\{u_n\}_{n\geq 1}$ be defined by induction as nonnegative R-solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x,t,\nabla u_1)) = \mu \text{ in } Q, \\ u_1 = 0 \text{ on } \partial\Omega \times (0,T), \\ u_1(0) = u_0 \text{ in } \Omega, \end{cases} \quad \begin{cases} (u_{n+1})_t - \operatorname{div}(A(x,t,\nabla u_{n+1})) = \mu - \lambda\mathcal{G}(u_n) \text{ in } Q, \\ u_{n+1} = 0 \text{ on } \partial\Omega \times (0,T), \\ u_{n+1}(0) = u_0 \text{ in } \Omega. \end{cases}$$

Thanks to Corollary 3.5.7 we can assume that $\{u_n\}$ is nondecreasing and satisfies for any $s > 0$ and $n \in \mathbb{N}$

$$\operatorname{meas}\{|u_n| \geq s\} \leq C_1 K_n s^{-p_c}, \tag{3.6.3}$$

where C_1 does not depend on s, n and

$$K_1 = (||u_0||_{1,\Omega} + |\mu|(Q))^{\frac{p+N}{N}},$$
$$K_{n+1} = (||u_0||_{1,\Omega} + |\mu|(Q) + \lambda||\mathcal{G}(u_n)||_{1,Q})^{\frac{p+N}{N}},$$

for any $n \geq 1$.Take $\varepsilon = \lambda + |\mu|(Q) + ||u_0||_{L^1(\Omega)} \leq 1$. Denoting by C_i some constants independent on n, ε, there holds $K_1 \leq C_2\varepsilon$, and for $n \geq 1$,

$$K_{n+1} \leq C_3\varepsilon(||\mathcal{G}(u_n)||_{1,Q}^{1+\frac{p}{N}} + 1).$$

From (3.6.1) and (3.6.3), we find

$$\|\mathcal{G}(u_n)\|_{L^1(Q)} \leq |Q|\, G(2) + \int_{\{u_n|\geq 2\}|} G(|u_n|)dxdt \leq |Q|\, G(2) + C_4 K_n \int_2^\infty G(s)\, s^{-1-p_c} ds.$$

Thus, $K_{n+1} \leq C_5\varepsilon(K_n^{1+\frac{p}{N}} + 1)$. Therefore, if ε is small enough, $\{K_n\}$ is bounded. Then, again from (3.6.1) and the relation $|\mathcal{G}(x,t,u_n)| \leq G(|u_n|)$ we verify that $\{\mathcal{G}(u_n)\}$ converges. Then by Theorem 3.2.1, up to a subsequence, $\{u_n\}$ converges to a R-solution u of (3.2.5).■

3.6.2 General case with absorption terms

In the sequel we assume that $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ does not depend on t. We recall a result obtained in [54, 17] in the elliptic case :

Theorem 3.6.2 *Let Ω be a bounded domain of $\mathbb{R}^N$ and $p < N$. Assume that $A : \Omega\times\mathbb{R}^N \to \mathbb{R}$ satisfies (3.1.6),(3.1.7). Let $\omega \in \mathfrak{M}_b(\Omega)$ with compact support in Ω. Suppose that u_n is a solution of problem*

$$\begin{cases} -div(A(x,\nabla u_n)) = \varphi_n * \omega \qquad in\ \Omega, \\ u_n = 0 \qquad on\ \partial\Omega, \end{cases}$$

where $\{\varphi_n\}$ is a sequence of mollifiers in $\mathbb{R}^N$. Then, up to subsequence, u_n converges a.e in Ω to a renormalized solution u of

$$\begin{cases} -div(A(x,\nabla u)) = \omega & in\ \Omega, \\ u = 0 \quad on\ \partial\Omega, \end{cases}$$

in the elliptic sense of [32], satisfying

$$-\kappa \mathbf{W}_{1,p}^{2diam(\Omega)}[\omega^-] \leqq u \leqq \kappa \mathbf{W}_{1,p}^{2diam(\Omega)}[\omega^+] \tag{3.6.4}$$

where κ is a constant which only depends of N, p, c_1, c_2.

Next we give a general result in case of absorption terms :

Theorem 3.6.3 *Let $p < N$, $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}$ satisfy (3.1.6) and (3.1.7), and $\mathcal{G} : Q \times \mathbb{R} \longmapsto \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto \mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. (x,t) in Q. Let $\mu_1, \mu_2 \in \mathfrak{M}_b^+(Q)$ such that there exist $\omega_n \in \mathfrak{M}_b^+(\Omega)$ and nondecreasing sequences $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ in $\mathfrak{M}_b^+(Q)$ with compact support in Q, converging to μ_1, μ_2, respectively in the narrow topology, and*

$$\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi_{(0,T)}, \qquad \mathcal{G}((n + \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega_n])) \in L^1(Q),$$

where the constant κ is given at Theorem 3.6.2. Let $u_0 \in L^1(\Omega)$, and $\mu = \mu_1 - \mu_2$. Then there exists a R-solution u of problem (3.1.5).

Moreover if $u_0 \in L^\infty(\Omega)$, and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$, for some $\gamma \in \mathfrak{M}_b^+(\Omega)$, then a.e. in Q,

$$|u(x,t)| \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\gamma](x) + ||u_0||_{\infty,\Omega}. \tag{3.6.5}$$

For proving this result, we need two Lemmas :

Lemma 3.6.4 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 3.6.3 and $\mathcal{G} \in L^\infty(Q \times \mathbb{R})$ and κ be the constant in Theorem 3.6.2. For $i = 1,2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, and $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in \mathfrak{M}_b^+(Q)$ with compact support in Q, $\gamma \in \mathfrak{M}_b^+(\Omega)$ with compact support in Ω such that $\lambda_i \leq \gamma \otimes \chi_{(0,T)}$. Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in Q. Then, there exist R-solutions u, u_1, u_2, to problems*

$$u_t - div(A(x,\nabla u)) + \mathcal{G}(u) = \lambda_1 - \lambda_2 \quad in\ Q, \qquad u = 0 \quad on\ \partial\Omega \times (0,T), \qquad u(0) = u_{0,1} - u_{0,2}, \tag{3.6.6}$$

$$(u_i)_t - div(A(x,\nabla u_i)) + \mathcal{G}(u_i) = \lambda_i \quad in\ Q, \qquad u_i = 0 \quad on\ \partial\Omega \times (0,T), \qquad u_i(0) = u_{0,i}, \tag{3.6.7}$$

relative to decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that a.e. in Q,

$$-||u_{0,2}||_{\infty,\Omega} - \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\gamma](x) \leq -u_2(x,t) \leq u(x,t) \leq u_1(x,t) \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\gamma](x) + ||u_{0,1}||_{\infty,\Omega}, \tag{3.6.8}$$

and

$$\int_Q |\mathcal{G}(u)| \leq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}\right), \quad \text{and} \quad \int_Q \mathcal{G}(u_i) \leq \lambda_i(Q) + ||u_{0,i}||_{1,\Omega}, \quad i = 1,2. \tag{3.6.9}$$

Furthermore, assume that $\mathcal{H}, \mathcal{K}$ have the same properties as $\mathcal{G}$, and $\mathcal{H}(x,t,s) \leq \mathcal{G}(x,t,s) \leq \mathcal{K}(x,t,s)$ for any $s \in (0,+\infty)$ and a.e. in Q. Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to $\mathcal{H}, \mathcal{K}$ with data λ_i, such that $u_i(\mathcal{H}) \geq u_i \geq u_i(\mathcal{K})$, $i = 1,2$.

Assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega_i, u_{0,i,2}), (\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, and $\varphi_n = \varphi_{1,n}\varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} * \gamma$, and for $i = 1,2$, $u_{0,i,n} = \varphi_{2,n} * u_{0,i}$,

$$\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \operatorname{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},$$

where $f_{i,n} = \varphi_n * f_i$, $g_{i,n} = \varphi_n * g_i$, $h_{i,n} = \varphi_n * h_i$, $\lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$, and

$$\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \operatorname{div}(g_n) + (h_n)_t + \lambda_{s,n},$$

where $f_n = f_{1,n} - f_{2,n}$, $g_n = g_{1,n} - g_{2,n}$, $h_n = h_{1,n} - h_{2,n}$, $\lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$. Then for n large enough, $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q)$, $\gamma_n \in C_c^\infty(\Omega)$. Thus there exist unique solutions $u_n, u_{i,n}, w_n$, $i = 1,2$, of problems

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,\nabla u_n)) + \mathcal{G}(u_n) = \lambda_{1,n} - \lambda_{2,n} & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,1,n} - u_{0,2,n} \quad \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \operatorname{div}(A(x,\nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \lambda_{i,n} & \text{in } Q, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{i,n}(0) = u_{0,i,n} \quad \text{in } \Omega, \end{cases}$$

$$-\operatorname{div}(A(x,\nabla w_n)) = \gamma_n \quad \text{in } \Omega, \qquad w_n = 0 \quad \text{on } \partial\Omega,$$

Moreover, as in the Proof of Theorem 3.2.2, (i), there holds

$$\int_Q |\mathcal{G}(u_n)| \leq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}\right), \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}, \quad i = 1,2.$$

By Proposition 3.4.10, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converge to some (u, u_1, u_2), *a.e.* in Q. Since $\mathcal{G}$ is bounded, in particular, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ and $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, (3.6.9) is satisfied. Moreover $\{\lambda_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$ and $\{\lambda_n - \mathcal{G}(u_n), f_n - \mathcal{G}(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$ are approximations of $(\lambda_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i})$ and $(\lambda_1 - \lambda_2 - \mathcal{G}(u), f - \mathcal{G}(u), g, h, \lambda_s, u_{0,1} - u_{0,2})$, in the sense of Theorem 3.2.1. Thus, we can find (different) subsequences converging *a.e.* to u, u_1, u_2, R-solutions of (3.6.6) and (3.6.7). Furthermore, from Theorem 3.6.2, up to a subsequence, $\{w_n\}$ converges *a.e.* in Q to a renormalized solution of

$$-\operatorname{div}(A(x,\nabla w)) = \gamma \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

such that $w \leqq \kappa \mathbf{W}_{1,p}^{2D}[\gamma]$ *a.e.* in Ω. Hence, we get the inequality (3.6.8). The other conclusions follow in the same way ■

Lemma 3.6.5 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 3.6.3 and κ be the constant in Theorem 3.6.2 . For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, $\lambda_i \in \mathfrak{M}_b^+(Q)$ with compact support in Q, and $\gamma \in \mathfrak{M}_b^+(\Omega)$ with compact support in Ω, such that*

$$\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \qquad \mathcal{G}((||u_{0,i}||_{\infty,\Omega} + \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}\gamma)) \in L^1(Q). \tag{3.6.10}$$

Then, there exist R-solutions u, u_1, u_2 of the problems (3.6.6) and (3.6.7), respectively relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, satifying (3.6.8) and (3.6.9).

Moreover, assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1}$. Then, one can find solutions $u_i(\omega_i, u_{0,i,2})$, $u_i(\theta_i, u_{0,i,1})$, corresponding with $(\omega_i, u_{0,i,2})$, $(\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1})$.

Proof. From Lemma 3.6.4 there exist R-solutions u_n, $u_{i,n}$ to problems

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,1} - u_{0,2} \quad \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i & \text{in } Q, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_{i,n}(0) = u_{0,i}, \quad \text{in } \Omega, \end{cases}$$

relative to the decompositions $(f_1 - f_2 - T_n(\mathcal{G}(u_n)), g_1 - g_2, h_1 - h_2)$, $(f_i - T_n(\mathcal{G}(u_{i,n})), g_i, h_i)$; and they satisfy

$$\begin{aligned} -||u_{0,2}||_{\infty,\Omega} - \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\gamma](x) &\leq -u_{2,n}(x,t) \leq u_n(x,t) \\ &\leq u_{1,n}(x,t) \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\gamma](x) + ||u_{0,1}||_{\infty,\Omega}, \end{aligned} \tag{3.6.11}$$

$$\int_Q |T_n(\mathcal{G}(u_n))| \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i}||_{1,\Omega}), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) \leq \lambda_i(Q) + ||u_{0,i}||_{1,\Omega}.$$

As in Lemma 3.6.4, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converges *a.e.* in Q to $\{u, u_1, u_2\}$ for which (3.6.8) is satisfied *a.e.* in Q. From (3.6.10), (3.6.11) and the dominated convergence Theorem, we deduce that $\{T_n(\mathcal{G}(u_n))\}$ converges to $\mathcal{G}(u)$ and $\{T_n(\mathcal{G}(u_{i,n}))\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, from Theorem 3.2.1, u and u_i are respective R-solutions of (3.6.6) and (3.6.7) relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, and (3.6.8) and (3.6.9 hold. The last statement follows from the same assertion in Lemma 3.6.4. ■

Proof of Theorem 3.6.3. By Proposition 3.3.2, for $i = 1, 2$, there exist $f_{i,n}, f_i \in L^1(Q)$, $g_{i,n}, g_i \in (L^{p'}(Q))^N$ and $h_{i,n}, h_i \in X$, $\mu_{i,n,s}, \mu_{i,s} \in \mathfrak{M}_s^+(Q)$ such that

$$\mu_i = f_i - \operatorname{div} g_i + (h_i)_t + \mu_{i,s}, \qquad \mu_{i,n} = f_{i,n} - \operatorname{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},$$

and $\{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\}$ strongly converge to f_i, g_i, h_i in $L^1(Q)$, $(L^{p'}(Q))^N$ and X respectively, and $\{\mu_{i,n}\}, \{\mu_{i,n,s}\}$ converge to $\mu_i, \mu_{i,s}$ (strongly) in $\mathfrak{M}_b(Q)$, and

$$||f_{i,n}||_{1,Q} + ||g_{i,n}||_{p',Q} + ||h_{i,n}||_X + \mu_{i,n,s}(\Omega) \leqq 2\mu(Q).$$

By Lemma 3.6.5, there exist R-solutions u_n, $u_{i,n}$ to problems

$$\begin{cases} (u_n)_t - \text{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = T_n(u_0) \quad \text{in } \Omega, \end{cases}$$

$$\begin{cases} (u_{i,n})_t - \text{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \mu_{i,n} & \text{in } Q, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{i,n}(0) = T_n(u_0^{\pm}) \quad \text{in } \Omega, \end{cases}$$

for $i = 1, 2$, relative to the decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that $\{u_{i,n}\}$ is nonnegative and nondecreasing, and $-u_{2,n} \leqq u_n \leqq u_{1,n}$; and

$$\int_Q |\mathcal{G}(u_n)| dxdt, \int_Q \mathcal{G}(u_{i,n}) dxdt \leqq \mu_1(Q) + \mu_2(Q) + ||u_0||_{1,\Omega}. \tag{3.6.12}$$

As in the proof of Lemma 3.6.5, up to a common subsequence $\{u_n, u_{1,n}, u_{2,n}\}$ converge *a.e.* in Q to $\{u, u_1, u_2\}$. Since $\{\mathcal{G}(u_{i,n})\}$ is nondecreasing, and nonnegative, from the monotone convergence Theorem and (5.1.6), we obtain that $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$, $i = 1, 2$. Finally, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$, since $|\mathcal{G}(u_n)| \leqq \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n})$. Thus, we can see that

$$\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_0)\}$$

is an approximation of $(\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)$, in the sense of Theorem 3.2.1. Therefore, u is a R-solution of (3.1.5), and (3.6.5) holds if $u_0 \in L^\infty(\Omega)$ and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$ and some $\gamma \in \mathfrak{M}_b^+(\Omega)$. ■

As a consequence we prove Theorem 3.2.3. We use the following result of [17] :

Proposition 3.6.6 (see [17]) *Let $q > p-1$, $\alpha > 0$and $\nu \in \mathfrak{M}_b^+(\Omega)$. If ν does not charge the sets of zero $Cap_{\mathbf{G}_p, \frac{q}{q+1-p}}$-capacity, there exists a nondecreasing sequence $\{\nu_n\} \subset \mathfrak{M}_b^+(\Omega)$ with compact support in Ω which converges to ν strongly in $\mathfrak{M}_b(\Omega)$ and such that $\mathbf{W}_{1,p}^R[\nu_n] \in L^q(\mathbb{R}^N)$, for any $n \in \mathbb{N}$, $R > 0$.*

Proof of Theorem 3.2.3. Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and $\mu \in \mathfrak{M}_b(Q)$ such that $|\mu| \leq \omega \otimes F$, where $F \in L^1((0,T))$ and ω does not charge the sets of zero $\text{Cap}_{\mathbf{G}_p, \frac{q}{q+1-p}}$-capacity. From Proposition 3.6.6, there exists a nondecreasing sequence $\{\omega_n\} \subset \mathfrak{M}_b^+(\Omega)$ with compact support in Ω which converges to ω, strongly in $\mathfrak{M}_b(\Omega)$, such that $\mathbf{W}_{1,p}^{2diam(\Omega)}[\omega_n] \in L^q(\mathbb{R}^N)$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-, \tag{3.6.13}$$

and $\mu^+, \mu^- \leq \omega \otimes F$. We set

$$Q_n = \{(x,t) \in \Omega \times (\frac{1}{n}, T - \frac{1}{n}) : d(x, \partial\Omega) > \frac{1}{n}\}, \qquad F_n = T_n(\chi_{(\frac{1}{n}T-\frac{1}{n})}F), \tag{3.6.14}$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \qquad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}. \tag{3.6.15}$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences with compact support in Q, and $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $(n + \kappa \mathbf{W}_{1,p}^{2diam(\Omega)}[\tilde{\omega}_n])^q \in L^1(Q)$. Besides, $\omega_n \otimes F_n$ converges to $\omega \otimes F$ strongly in $\mathfrak{M}_b(Q)$: indeed we easily check that

$$||\omega_n \otimes F_n - \omega \otimes F||_{\mathfrak{M}_b(Q)} \leq ||F_n||_{L^1((0,T))} ||\omega_n - \omega||_{\mathfrak{M}_b(\Omega)} + ||\omega||_{\mathfrak{M}_b(\Omega)} ||F_n - F||_{L^1((0,T))}$$

Observe that for any measures $\nu, \theta, \eta \in \mathfrak{M}_b(Q)$, there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leq |\theta - \eta|,$$

hence $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ converge to μ_1, μ_2 respectively in $\mathfrak{M}_b(Q)$. Therefore, the result follows from Theorem 3.6.3. ■

Remark 3.6.7 *Let $\mathcal{G} : Q \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto \mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. (x,t) in Q. Let $\mu \in \mathcal{M}_b(Q), f \in L^1(Q), u_0 \in L^1(\Omega)$ and $\omega \in \mathcal{M}_b^+(\Omega)$ such that (3.2.6) holds.*

If $\omega(\{x : \mathbf{W}_{1,p}^{2diam(\Omega)}[\omega](x) = \infty\}) = 0$, then, (3.1.5) has a R-solution with data $(f + \mu, u_0)$. The proof is similar to the one of Theorem 3.2.3, after replacing ω_n by $\chi_{\mathbf{W}_{1,p}^{2diam(\Omega)}[\omega] \leq n}\omega$. Note that $\omega(\{x : \mathbf{W}_{1,p}^{2diam(\Omega)}[\omega](x) = \infty\}) = 0$ if and only if ω is diffuse, see [46].

Remark 3.6.8 *As in [17], from Theorem 3.6.3, we can extend Theorem 3.2.3 given for $\mathcal{G}(u) = |u|^{q-1}u$, to the case of a function $\mathcal{G}(x,t,.)$, odd for a.e. $(x,t) \in Q$, such that*

$$|\mathcal{G}(x,t,u)| \leq G(|u|), \qquad \int_1^\infty G(s) s^{-q-1} ds < \infty,$$

where G is a nondecreasing continuous, under the condition that ω does not charge the sets of zero $Cap_{\mathbf{G}_p, \frac{q}{q-p+1}, 1}$-capacity, where for any Borel set $E \subset \mathbb{R}^N$,

$$Cap_{\mathbf{G}_p, \frac{q}{q-p+1}, 1}(E) = \inf\{||\varphi||_{L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N), \quad \mathbf{G}_p * \varphi \geq \chi_E\}$$

where $L^{\frac{q}{q-p+1}, 1}(\mathbb{R}^N)$ is the Lorentz space of order $(\frac{q}{q-p+1}, 1)$.

In case $\mathcal{G}$ is of exponential type, we introduce the notion of maximal fractional operator, defined for any $\eta \geq 0$, $R > 0$, $x_0 \in \mathbb{R}^N$ by

$$\mathbf{M}_{p,R}^\eta[\omega](x_0) = \sup_{t \in (0,R)} \frac{\omega(B(x_0,t))}{t^{N-p} h_\eta(t)}, \qquad \text{where } h_\eta(t) = \inf((-\ln t)^{-\eta}, (\ln 2)^{-\eta})).$$

We obtain the following :

Theorem 3.6.9 *Let $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}$ satisfy (3.1.6) and (3.1.7). Let $p < N$ and $\tau > 0, \beta > 1, \mu \in \mathfrak{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that $|\mu| \leq \omega \otimes F$, with $\omega \in \mathfrak{M}_b^+(\Omega)$, $F \in L^1((0,T))$ be nonnegative. Assume that one of the following assumptions is satisfied :*

(i) $||F||_{L^\infty((0,T))} \leq 1$ *and for some* $M_0 = M_0(N, p, \beta, \tau, c_3, c_4, \mathrm{diam}\Omega)$,

$$||\mathbf{M}_{p,2\mathrm{diam}(\Omega)}^{\frac{p-1}{\beta'}}[\omega]||_{L^\infty(\mathbb{R}^N)} < M_0, \tag{3.6.16}$$

(ii) *There exists $\beta_0 > \beta$ such that* $\mathbf{M}_{p,2diam(\Omega)}^{\frac{p-1}{\beta_0'}}[\omega] \in L^\infty(\mathbb{R}^N)$.

Then there exists a R-solution to the problem

$$\begin{cases} u_t - div(A(x, \nabla u)) + (e^{\tau|u|^\beta} - 1)\mathrm{sign}(u) = f + \mu & in\ Q, \\ u = 0 \qquad on\ \partial\Omega \times (0,T), \\ u(0) = u_0 \qquad in\ \Omega. \end{cases} \tag{3.6.17}$$

For the proof we use the following result of [17] :

Proposition 3.6.10 (see [17], Theorem 2.4) *Suppose $1 < p < N$. Let $\nu \in \mathfrak{M}_b^+(\Omega)$, $\beta > 1$, and $\delta_0 = ((12\beta)^{-1})^\beta p \ln 2$. There exists $C = C(N, p, \beta, \mathrm{diam}\Omega)$ such that, for any $\delta \in (0, \delta_0)$,*

$$\int_\Omega \exp\left(\delta \frac{(\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\nu])^\beta}{||\mathbf{M}_{p,2\mathrm{diam}(\Omega)}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}}}\right) dx \leq \frac{C}{\delta_0 - \delta}.$$

Proof of Theorem 3.6.9. Let Q_n be defined at (3.6.14), and $\omega_n = \omega\chi_{\Omega_n}$, where $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$. We still consider $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$ as in (3.6.13), (3.6.15).
Case 1 : Assume that $||F||_{L^\infty((0,T))} \leq 1$ and (3.6.16) holds. We have $\mu_{1,n}, \mu_{2,n} \leq n\chi_\Omega + \omega$. For any $\varepsilon > 0$, there exists $c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, \mathrm{diam}\Omega) > 0$ such that

$$(n + \kappa\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[n\chi_\Omega + \omega])^\beta \leq c_\varepsilon n^{\frac{\beta p}{p-1}} + (1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega])^\beta$$

a.e. in Ω. Thus,

$$\exp\left(\tau(n + \kappa\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[n\chi_\Omega + \omega])^\beta\right) \leq \exp\left(\tau c_\varepsilon n^{\frac{\beta p}{p-1}}\right) \exp\left(\tau(1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega])^\beta\right)$$

If (3.6.16) holds with $M_0 = \left(\delta_0/\tau\kappa^\beta\right)^{(p-1)/\beta}$ then we can chose ε such that

$$\tau(1+\varepsilon)\kappa^\beta||\mathbf{M}_{p,2\mathrm{diam}(\Omega)}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}} < \delta_0.$$

From Proposition 3.6.10, we get $\exp(\tau(1+\varepsilon)\kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega])^\beta) \in L^1(\Omega)$, which implies $\exp(\tau(n + \kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[n\chi_\Omega + \omega])^\beta) \in L^1(\Omega)$ for all n. We conclude from Theorem 3.6.3.

Case 2 : Assume that there exists $\varepsilon > 0$ such that $\mathbf{M}_{p,2diam(\Omega)}^{(p-1)/(\beta+\varepsilon)'}[\omega] \in L^{\infty}(\mathbb{R}^N)$. Now we use the inequality $\mu_{1,n}, \mu_{2,n} \leq n(\chi_\Omega + \omega)$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $c_{\varepsilon,n} > 0$ such that

$$(n + \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[n(\chi_\Omega + \omega)])^\beta \leq c_{\varepsilon,n} + \varepsilon(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^{\beta_0}$$

Thus, from Proposition 3.6.10 we get $\exp(\tau(n + \kappa^\beta \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[n(\chi_\Omega + \omega)])^\beta) \in L^1(\Omega)$ for all n. We conclude from Theorem 3.6.3. ■

3.6.3 Equations with source term

As a consequence of Theorem 3.6.3, we get a first result for problem (3.1.1) :

Corollary 3.6.11 *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy (3.1.6) and (3.1.7). Let $u_0 \in L^\infty(\Omega)$, and $\mu \in \mathfrak{M}_b(Q)$ such that $|\mu| \leq \omega \otimes \chi_{(0,T)}$ for some $\omega \in \mathfrak{M}_b^+(\Omega)$. Then there exist a R-solution u of (3.1.1), such that*

$$|u(x,t)| \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega](x) + ||u_0||_{\infty,\Omega}, \qquad \text{for a.e. } (x,t) \in Q, \tag{3.6.18}$$

where κ is defined at Theorem 3.6.2.

Proof. Let $\{\phi_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Since $\{\phi_n\mu^+\}, \{\phi_n\mu^-\}$ are nondecreasing sequences, the result follows from Theorem 3.6.3. ■

Our proof of Theorem 3.2.4 is based on a property of Wölf potentials :

Theorem 3.6.12 (see [54]) *Let $q > p-1$, $0 < p < N$, $\omega \in \mathfrak{M}_b^+(\Omega)$. If for some $\lambda > 0$,*

$$\omega(E) \leq \lambda Cap_{\mathbf{G}_p, \frac{q}{p-q+1}}(E) \qquad \text{for any compact set } E \subset \mathbb{R}^N, \tag{3.6.19}$$

then $(\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega])^q \in L^1(\Omega)$, and there exists $M = M(N,p,q,\mathrm{diam}(\Omega))$ such that, a.e. in Ω,

$$\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}\left[\left(\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega]\right)^q\right] \leq M\lambda^{\frac{q-p+1}{(p-1)^2}} \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega] < \infty. \tag{3.6.20}$$

We deduce the following :

Lemma 3.6.13 *Let $\omega \in \mathfrak{M}_b^+(\Omega)$, and $b \geq 0$ and $K > 0$. Suppose that $\{u_m\}_{m\geq 1}$ is a sequence of nonnegative functions in Ω that satisfies*

$$u_1 \leq K\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega] + b, \qquad u_{m+1} \leq K\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[u_m^q + \omega] + b \qquad \forall m \geq 1.$$

Assume that ω satisfies (3.6.19) for some $\lambda > 0$. Then there exist λ_0 and b_0, depending on N, p, q, K, and $\mathrm{diam}(\Omega)$, such that, if $\lambda \leq \lambda_0$ and $b \leq b_0$, then $\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mu] \in L^q(\Omega)$ and for any $m \geq 1$,

$$u_m \leq 2\beta_p K\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega] + 2b, \qquad \beta_p = \max(1, 3^{\frac{2-p}{p-1}}). \tag{3.6.21}$$

Proof. Clearly, (3.6.21) holds for $m = 1$. Now, assume that it holds at the order m. Then

$$u_m^q \leq 2^{q-1}(2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega])^q + 2^{q-1}(2b)^q.$$

Using (3.6.20) we get

$$\begin{aligned} u_{m+1} &\leq K\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}\left[2^{q-1}(2\beta_p)^q K^q (\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega])\right)^q + 2^{q-1}(2b)^q + \omega\Big] + b \\ &\leq \beta_p K\left(A_1 \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}\left[(\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega])^q\right] + \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[(2b)^q] + \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega]\right) + b \\ &\leq \beta_p K(A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega] + \beta_p K \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[(2b)^q] + b \\ &= \beta_p K(A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega] + A_2 b^{\frac{q}{p-1}} + b, \end{aligned}$$

where M is as in (3.6.20) and

$$A_1 = \left(2^{q-1}(2\beta_p)^q K^q\right)^{1/(p-1)}, A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1} (2\text{diam}(\Omega))^{p'}.$$

Thus, (3.6.21) holds for $m = n + 1$ if we prove that

$$A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} \leq 1 \text{ and } A_2 b^{\frac{q}{p-1}} \leq b,$$

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}} \text{ and } b \leq A_2^{-\frac{p-1}{q-p+1}}.$$

Therefore, we obtain the result with $\lambda_0 = (A_1 M)^{-(p-1)^2/(q-p+1)}$ and $b_0 = A_2^{-(p-1)/(q-p+1)}$.
■

Proof of Theorem 3.2.4. From Corollary 3.5.7 and 3.6.11, we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m\geq 1}$ defined in the following way : u_1 is a R-solution of (3.1.1), and u_{m+1} is a nonnegative R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = u_m^q + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 \quad \text{in } \Omega. \end{cases}$$

Setting $\overline{u}_m = \sup_{t\in(0,T)} u_m(t)$ for all $m \geq 1$, there holds

$$\overline{u}_1 \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega] + ||u_0||_{\infty,\Omega}, \qquad \overline{u}_{m+1} \leq \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\overline{u}_m^q + \omega] + ||u_0||_{\infty,\Omega} \qquad \forall m \geq 1.$$

From Lemma 3.6.13, we can find $\lambda_0 = \lambda_0(N, p, q, diam\Omega)$ and $b_0 = b_0(N, p, q, \text{diam}\Omega)$ such that if (3.2.8) is satisfied with λ_0 and b_0, then

$$u_m \leq \overline{u}_m \leq 2\beta_p \kappa \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\omega] + 2||u_0||_{\infty,\Omega} \qquad \forall m \geq 1. \tag{3.6.22}$$

Thus $\{u_m\}$ converges *a.e.* in Q and in $L^1(Q)$ to some function u, for which (3.2.10) is satisfied in Ω with $c = 2\beta_p\kappa$. Finally, one can apply Theorem 3.2.1 to the sequence of measures $\{u_m^q + \mu\}$, and obtain that u is a R-solution of (3.2.9). ■

Next we consider the exponential case.

Theorem 3.6.14 *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy (3.1.6) and (3.1.7). Let $\tau > 0, l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Set*

$$\mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \qquad \forall s \in \mathbb{R}. \tag{3.6.23}$$

Let $\mu \in \mathfrak{M}_b^+(Q)$, $\omega \in \mathfrak{M}_b^+(\Omega)$ such that $\mu \leq \chi_{(0,T)} \otimes \omega$. Then, there exist b_0 and M_0 depending on N, p, β, τ, l and $\mathrm{diam}\Omega$, such that if

$$||\mathbf{M}_{p,2\mathrm{diam}(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq M_0, \qquad ||u_0||_{\infty,\Omega} \leq b_0,$$

the problem

$$\begin{cases} u_t - div(A(x, \nabla u)) = \mathcal{E}(\tau u^\beta) + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 \quad \text{in } \Omega \end{cases} \tag{3.6.24}$$

admits nonnegative R- solution u, which satisfies, a.e. in Q, for some c, depending on N, p, c_3, c_4

$$u(x, t) \leq c\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega](x) + 2b_0. \tag{3.6.25}$$

For the proof we first recall an approximation property, which is a consequence of [47, Theorem 2.5] :

Theorem 3.6.15 *Let $\tau > 0$, $b \geq 0$, $K > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Let $\mathcal{E}$ be defined by (3.6.23). Let $\{v_m\}$ be a sequence of nonnegative functions in Ω such that, for some $K > 0$,*

$$v_1 \leq K\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mu] + b, \qquad v_{m+1} \leq K\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mathcal{E}(\tau v_m^\beta) + \mu] + b, \quad \forall m \geq 1.$$

Then, there exist b_0 and M_0, depending on N, p, β, τ, l, K and $\mathrm{diam}\Omega$ such that if $b \leq b_0$ and

$$||\mathbf{M}_{p,2\mathrm{diam}(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{\infty,\mathbb{R}^N} \leq M_0, \tag{3.6.26}$$

then, setting $c_p = 2\max(1, 2^{\frac{2-p}{p-1}})$,

$$\exp(\tau(Kc_p\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mu] + 2b_0)^\beta) \in L^1(\Omega),$$

$$v_m \leq Kc_p\mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mu] + 2b_0, \quad \forall m \geq 1. \tag{3.6.27}$$

Proof of Theorem 3.6.14. From Corollary 3.5.7 and 3.6.11 we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m\geq1}$ defined in the following way : u_1 is a R-solution of problem (3.1.1), and by induction, u_{m+1} is a R-solution of

$$\begin{cases} (u_{m+1})_t - \mathrm{div}(A(x, \nabla u_{m+1})) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_{m+1}(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{3.6.28}$$

And, setting $\overline{u}_m = \sup_{t\in(0,T)} u_m(t)$, there holds

$$\overline{u}_1 \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega]+||u_0||_{\infty,\Omega}, \qquad \overline{u}_{m+1} \leq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\mathcal{E}(\tau\overline{u}_m^{\beta})+\omega]+||u_0||_{\infty,\Omega}, \qquad \forall m \geq 1.$$

Thus, from Theorem 3.6.15, there exist $b_0 \in (0,1]$ and $M_0 > 0$ depending on N, p, β, τ, l and diamΩ such that, if (3.6.26) holds, then (3.6.27) is satisfied with $v_m = \overline{u}_m$. As a consequence, u_m is well defined. Thus, $\{u_m\}$ converges $a.e.$ in Q to some function u, for which (3.6.25) is satisfied in Ω. Furthermore, $\left\{\mathcal{E}(\tau u_m^{\beta})\right\}$ converges to $\mathcal{E}(\tau u^{\beta})$ in $L^1(Q)$. Finally, one can apply Theorem 3.2.1 to the sequence of measures $\left\{\mathcal{E}(\tau u_m^{\beta}) + \mu\right\}$, and obtain that u is a R-solution of (3.6.24). ∎

Remark 3.6.16 *In [47, Theorem 1.1], when div$(A(x,\nabla u)) = \Delta_p u$, we showed that there exist $M = M(N, p, \beta, \tau, l, diam(\Omega))$ such that if*

$$||\mathbf{M}_{p,2diam(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^{\infty}(\mathbb{R}^N)} \leqq M,$$

then the problem

$$\begin{cases} -\Delta_p v = \mathcal{E}(\tau v^{\beta}) + \omega & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{3.6.29}$$

has a renormalized solution in the sense of [17]. We claim the following :

Let div$(A(x,\nabla u)) = \Delta_p u$ and $u_0 \equiv 0$. If (3.6.29) has a renormalized solution v and ω is diffuse, then the problem (3.6.24) *in Theorem 3.6.14 admits a R-solution u, satisfying $u(x,t) \leqq v(x)$ a.e in Q.*

Indeed, since ω is diffuse, there holds $\mu \in \mathcal{M}_0(Q)$. Otherwise, for any measure $\eta \in \mathcal{M}_0(Q)$ the problem

$$\begin{cases} u_t - \Delta_p u = \eta & in\ Q, \\ u = 0 & on\ \partial\Omega \times (0,T), \\ u = 0 & in\ \Omega, \end{cases}$$

has a (unique) R-solution, and the comparison principle is valid, see [50]. Thus, as in the proof of Theorem 3.6.14, we can construct a ***unique*** *sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m\geqq 1}$, defined in the following way : u_1 is a R-solution of problem (3.1.1) and satisfies $u_1 \leqq v$ a.e in Q ; and by induction, u_{m+1} is a R-solution of (3.6.28) and satisfies $u_{m+1} \leqq v$ a.e in Q. Then $\left\{\mathcal{E}(\tau u_m^{\beta})\right\}$ converges to $\mathcal{E}(\tau u^{\beta})$ in $L^1(Q)$. Finally, $u := \lim_{n\to\infty} u_n$ is a solution of (3.6.24). Clearly, this claim is also valid for power source term.*

3.7 Appendix

Proof of Lemma 3.4.7. Let $\mathcal{J}$ be defined by (3.4.11). Let $\zeta \in C_c^1([0,T))$ with values in $[0,1]$, such that $\zeta_t \leq 0$, and $\varphi = \zeta\xi[J(S(v))]_l$. Clearly, $\varphi \in X \cap L^{\infty}(Q)$; we choose the pair of functions (φ, S) as test function in (3.4.2). Thanks to convergence properties of Steklov time-averages, we easily will obtain (3.4.15) if we prove that

$$\varliminf_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[J(S(v))]_l)_t S(v)) \geq -\int_Q \xi_t \overline{J}(S(v)).$$

We can write $-\int_Q (\zeta\xi[J(S(v))]_l)_t S(v) = F + G$, with

$$F = -\int_Q (\zeta\xi)_t [J(S(v))]_l S(v), \qquad G = -\int_Q \zeta\xi S(v)\frac{1}{l}\left(J(S(v))(x,t+l) - J(S(v))(x,t)\right).$$

Using (3.4.12) and integrating by parts we have

$$\begin{aligned} G &\geq -\int_Q \zeta\xi\frac{1}{l}\left(\mathcal{J}(S(v))(x,t+l) - \mathcal{J}(S(v))(x,t)\right) \\ &= -\int_Q \zeta\xi\frac{\partial}{\partial t}\left([\mathcal{J}(S(v))]_l\right) = \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l + \int_\Omega \zeta(0)\xi(0)[\mathcal{J}(S(v))]_l(0)dx \\ &\geq \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l, \end{aligned}$$

since $\mathcal{J}(S(v)) \geq 0$. Hence,

$$-\int_Q (\zeta\xi[j(S(v))]_l)_t S(v) \geq \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l + F = \int_Q (\zeta\xi)_t\left([\mathcal{J}(S(v))]_l - [J(S(v))]_l S(v)\right)$$

Otherwise, $\mathcal{J}(S(v))$ and $J(S(v)) \in C([0,T]\,; L^1(\Omega))$, thus $\{(\zeta\xi)_t\left([\mathcal{J}(S(u))]_l - [J(S(u))]_l S(u)\right)\}$ converges to $-(\zeta\xi)_t\overline{J}(S(u))$ in $L^1(Q)$ as $l \to 0$. Therefore,

$$\begin{aligned} \varliminf_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[J(S(v))]_l)_t S(v)) &\geq \lim_{\zeta\to 1}\left(-\int_Q (\zeta\xi)_t\overline{J}(S(v))\right) \\ &\geq \lim_{\zeta\to 1}\left(-\int_Q \zeta\xi_t\overline{J}(S(v))\right) = -\int_Q \xi_t\overline{J}(S(v)), \end{aligned}$$

which achieves the proof. ∎

Bibliographie

[1] Adams D. and Polking J., *The equivalence of two definitions of the capacity*, Proc. A.M.S., **37**, 529-534 (1973).

[2] F. Andreu,J.M. Mazon, S. Segura de Leon and J. Toledo *Existence and uniqueness for a degenerate parabolic equation with* L^1 *data*, Trans. Amer. Math. Soc. **351**, 285-306 (1999).

[3] Andreu F., Mazon J., Segura de Leon S. and Toledo J. ,*Quasilinear elliptic and parabolic equations in* L^1 *with nonlinear boundary conditions*, Adv. Math. Sci. Appl. **7**, 183-213 (1997).

[4] Andreianov B., Sbihi K., and Wittbold P., *On uniqueness and existence of entropy solutions for a nonlinear parabolic problem with absorption*, J. Evol. Equ. **8**, 449-490 (2008).

[5] Baras P. and Pierre M., *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier, **34**, 185-206 (1984).

[6] Baras P. and Pierre M., *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. I.H.P. **2**, 185-212 (1985).

[7] Baras P. and Pierre M., *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18** , 111-149 (1984).

[8] Di Benedetto E., *On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients,* Ann. Sc. Norm. Sup. Pisa, **13** , 487-535 (1996).

[9] E.D. Benedetto, *Degenerate Parabolic Equations*, Springer-Verlag (1993).

[10] E.D. Benedetto, M.A. Herrero, *On the Cauchy problem and initial trace for a degenerate parabolic equation*, Trans. Amer. Math. Soc. **314**, 187-223 (1989).

[11] E.D. Benedetto, M.A. Herrero, *Nonnegative solutions of the evolution p-Laplacian equation. Initial trace and Cauchy problem when* $1 < p < 2$, Arch. Rat. Mech. Anal. **111**, 225-290 (1990).

[12] E.D. Benedetto and Y.Z. Chen, *On the local behaviour of solutions of singular parabolic equation*, Arch. Rat. Mech. Anal. **107**, 293-324 (1989).

[13] Benilan P., Boccardo L., Gallouet T., Gariepy R., Pierre M. and Vázquez J., *An* $L1$-*theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22**, no. 2, 241-273 (1995).

[14] Bidaut-Véron M.F., *Removable singularities and existence for a quasilinear equation*, Adv. Nonlinear Studies **3**, 25-63 (2003).

[15] Bidaut-Véron M.F., *Necessary conditions of existence for an elliptic equation with source term and measure data involving the p-Laplacian,* E.J.D.E., Conference **08**, 23-34 (2002).

[16] Bidaut-Véron M.F., Chasseigne E. and Véron L. *Initial trace of solutions odf some quasilinear parabolic equations with absorption,* J. Funct. Anal. **193**, 140-205 (2002).

[17] M.F. Bidaut-Véron, N.Q. Hung and L. Véron , *Quasilinear Lane-Emden equations with absorption terms and measure data*,J. Math. Pures Appl, **102**, 315-337 (2014).

[18] D. Blanchard, *Truncations and monotonicity methods for parabolic equations*, Nonlinear Anal. T., M. A. **21**, 725-743 (1993).

[19] D. Blanchard, F. Murat, *Renormalized solutions of nonlinear parabolic equation with L^1 data : existence and uniqueness*, Proc. Roy. Soc. Edinburgh **127A**, 1153-1179 (1997).

[20] Blanchard D., Petitta F. and Redwane H., *Renormalized solutions of nonlinear parabolic equations with diffuse measure data, Manuscripta Math.* **141**, 601-635 (2013).

[21] D. Blanchard and A. Porretta , *Nonlinear parabolic equations with natural growth terms and measure initial data,* Ann.Scuola Norm. Su. Pisa Cl Sci. **30**, 583-622 (2001).

[22] D. Blanchard, A. Porretta, *Stefan problems with nonlinear diffusion and convection,* J. Diff. Equ. **210**, 383-428 (2005).

[23] L. Boccardo, T. Gallouet, *Nonlinear elliptic and parabolic equations involving measure data,* J Funct. Anal. **87**, 149-169 (1989).

[24] L. Boccardo and T. Gallouet, *Nonlinear elliptic equations with right-hand side measures,* Comm. Partial Diff. Equ. **17**, 641-655 (1992).

[25] L. Boccardo, T. Gallouet, L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data,* Ann. Inst. H. Poincaré Anal. non Lin. **13**, 539-555 (1996).

[26] L. Boccardo, A. Dall'Aglio, T. Gallouet and L. Orsina, *Nonlinear parabolic equations with measure data, J. Funct. Anal.* **147**, 237-258 (1996).

[27] L. Boccardo, F. Murat, J. Puel, *Résultats d'existence pour certains problèmes elliptiques quasilinéaires,* Ann. Scuola. Norm. Sup. Pisa **11**, 213-235 (1984).

[28] H. Brezis, A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions,* J.Math.Pures Appl. **62**, 73-97 (1983).

[29] X. Chen, Y. Qi, M. Wang, *Singular solutions of parabolic p-Laplacian with absorption*, T.A.M.S. **359**, 5653-5668, (2007).

[30] A. Dall'Aglio, L. Orsina, *Existence results for some nonlinear parabolic equations with nonregular data*, Diff. Int. Equ. **5**, 1335-1354 (1992).

[31] A. Dall'Aglio, L. Orsina, *Nonlinear parabolic equations with natural growth conditions and L^1*data, Nonlinear Anal. **27**, 59-73 (1996).

[32] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data,* Ann. Scuola Norm. Sup. Pisa, **28**, 741-808 (1999).

[33] J. Droniou, A. Porretta, A. Prignet, *Parabolic capacity and soft measures for nonlinear equations,* Potential Anal. **19**, 99-161 (2003).

[34] J. Droniou, A. Prignet, *Equivalence between entropy and renormalized solutions for parabolic equations with smooth data,* Nonlinear Diff Eq. Appl. **14**, 181-205 (2007).

[35] A. Gmira., *On quasilinear parabolic equations involving measure data*, Asymptotic Anal. **3**, 43-56 (1990).

[36] J. Heinonen, T. Kilpelainen, O. Martio*Nonlinear potential theory of degenerate elliptic equations*, Oxford Science Publications, (1993).

[37] S. Kamin, L.A. Peletier, *Source-type solutions of degenerate diffusion equations with absorption*, Isr. J. Math. **50**, 219-230 (1985).

[38] S. Kamin, L.A. Peletier, J.L. Vazquez, *Classification of singular solutions of a nonlinear heat equation*, Duke Math. J. **58**, 601-615 (1989).

[39] S. Kamin, J. L. Vazquez, *Singular solutions of some nonlinear parabolic equations,* J. Analyse Math. **59**, 51-74 (1992).

[40] T. Kilpeläinen, X. Xu, *0n the uniqueness problem for quasilinear elliptic equations involving measure,* Revista Matematica Iberoamericana. **12**, 461-475 (1996).

[41] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial boundary-value problems,* Proc. Royal Soc. Edinburg Sect A, **89**, 217-237 (1981).

[42] T. Leonori, F. Petitta, *Local estimates for parabolic equations with nonlinear gradient terms,* Calc. Var. Partial Diff. Equ. **42**, 153-187 (2011).

[43] F. Leoni, B. Pellacci, *Local estimates and global existence for strongly nonlinear parabolic equations with locally integrable data*, J. Evol. Equ. **6**, 113-144 (2006).

[44] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod et Gauthiers-Villars (1969).

[45] M. Marcus, L. Véron, *Initial trace of positive solutions of some nonlinear parabolic equations,* Comm. Partial Diff. Equ. **24**, 1445-1499 (1999).

[46] Mikkonen P., *On the Wolff potential and quasilinear elliptic equations involving measures,* Ann. Acad. Sci. Fenn. Ser AI, Math. Dissert. 104 (1996).

[47] Quoc-Hung Nguyen and L. Véron, *Quasilinear and Hessian equations with exponential reaction and measure data,* Archive for Rational Mechanics and Analysis, **214**, 235-267 (2014).

[48] F. Petitta, Asymptotic behavior of solutions for linear parabolic equations with general measure data, C. R. Acad. Sci. Paris, Ser. I **344** 571-576 (2007) .

[49] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data,* Ann. Math. Pura Appl. **187**, 563-604 (2008).

[50] F. Petitta, A. Ponce, A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ. **11**, 861-905 (2011).

[51] Pierre M., *Parabolic capacity and Sobolev spaces*, Siam J. Math. Anal. 14 (1983), 522-533.

[52] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations,* Ann. Mat. Pura Apll. **177**, 143-172 (1999).

[53] A. Porretta, *Nonlinear equations with natural growth terms and measure data,* E.J.D.E., Conference **9**, 183-202 (2002).

[54] N.C. Phuc, E.I. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type.* Ann. of Math. **168**, 859-914 (2008).

[55] A. Prignet, *Existence and uniqueness of "entropy" solutions of parabolic problems with L^1 data,* Nonlinear Anal. TMA **28**, 1943-1954 (1997).

[56] J. Rakotoson, *Some quasilinear parabolic equations*, Nonlinear Anal. **17**, 1163-1175 (1991).

[57] J. Rakotoson, *A compactness Lemma for quasilinear problems : application to parabolic equations*, J. Funct. Anal. **106**, 358-374 (1992).

[58] J. Rakotoson, *Generalized solutions in a new type of sets for problems with measures as data,* Diff. Int. Equ. **6**, 27-36 (1993).

[59] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier,**15**, 189-258 (1965).

[60] X. Xu, *On the initial boundary-value-problem for* $u_t - \mathrm{div}(|\nabla u|^{p-2} |\nabla u|) = 0$, Arch. Rat. Mech. Anal. **127**, 319-335 (1994).

[61] J. Zhao, *Source-type solutions of a quasilinear degenerate parabolic equation with absorption,* Chin. Ann. of MathB,**1**, 89-104 (1994).

[62] J. Zhao, *Source-type solutions of a degenerate quasilinear parabolic equations,* J. Diff. Equ. **92**, 179-198 (1991).

Chapitre 4

Potential estimates and quasilinear parabolic equations with measure data

Abstract

In this paper, we study the existence and regularity of the quasilinear parabolic equations :

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \mu$$

in $\mathbb{R}^{N+1}$, $\mathbb{R}^N \times (0,\infty)$ and a bounded domain $\Omega \times (0,T) \subset \mathbb{R}^{N+1}$. Here $N \geq 2$, the nonlinearity A fulfills standard growth conditions and B term is a continuous function and μ is a radon measure. Our first task is to establish the existence results with $B(u,\nabla u) = \pm |u|^{q-1}u$, for $q > 1$. We next obtain global weighted-Lorentz, Lorentz-Morrey and Capacitary estimates on gradient of solutions with $B \equiv 0$, under minimal conditions on the boundary of domain and on nonlinearity A. Finally, due to these estimates, we solve the existence problems with $B(u,\nabla u) = |\nabla u|^q$ for $q > 1$.

4.1 Introduction

In this article, we study a class of quasilinear parabolic equations :

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(x,t,u,\nabla u) + \mu \tag{4.1.1}$$

in $\mathbb{R}^{N+1}$ or $\mathbb{R}^N \times (0,\infty)$ or a bounded domain $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{N+1}$. Where $N \geq 2$, $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies

$$|A(x,t,\zeta)| \leq \Lambda_1 |\zeta| \quad \text{and} \tag{4.1.2}$$

$$\langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda \rangle \geq \Lambda_2 |\zeta - \lambda|^2, \tag{4.1.3}$$

for every $(\lambda, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, here Λ_1 and Λ_2 are positive constants, $B : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is also a Carathéodory function and μ is a Radon measure.

The existence and regularity theory, the Wiener criteria and Harnack inequalities, Blow-up at a finite time associated with above parabolic quasilinear operator was studied and developed intensely over the past 50 years, one can found in [58, 44, 30, 48, 49, 25, 50, 60, 83, 75, 73]. Moreover, we also refer to [19]-[22] for L^p-gradient estimates theory in non-smooth domains and [63] Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

First, we are specially interested in the existence of solutions to quasilinear parabolic equations with absorption, source terms and data measure :

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \tag{4.1.4}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \tag{4.1.5}$$

in $\mathbb{R}^{N+1}$ and

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \quad u(0) = \sigma \tag{4.1.6}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu, \quad u(0) = \sigma \tag{4.1.7}$$

in $\mathbb{R}^N \times (0,\infty)$ or a bounded domain $\Omega_T \subset \mathbb{R}^{N+1}$, where $q > 1$ and μ, σ are Radon measures.

The linear case $A(x,t,\nabla u) = \nabla u$ was studied in detail by Fujita, Brezis and Friedman, Baras and Pierre.

In [18], showed that if $\mu = 0$ and σ is a Dirac mass in Ω, the problem (4.1.6) in Ω_T (with Dirichlet boundary condition) admits a (unique) solution if and only if $q < (N+2)/N$. Then, optimal results had been considered in [5], for any $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$: there exists a (unique) solution of (4.1.6) in Ω_T if and only if μ, σ are absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{2/q},q'}$ (in Ω_T, Ω) respectively, for simplicity we write $\mu << \text{Cap}_{2,1,q'}$ and $\sigma << \text{Cap}_{\mathbf{G}_{2/q},q'}$, with q' is the conjugate exponent of q, i.e $q' = \frac{q}{q-1}$. Where these two capacities will be defined in section 2.

For source case, in [6], showed that for any $\mu \in \mathfrak{M}_b^+(\Omega_T)$ and $\sigma \in \mathfrak{M}_b^+(\Omega)$, the problem (4.1.7) in bounded domain Ω_T has a nonnegative solution if

$$\mu(E) \leq C\text{Cap}_{2,1,q'}(E) \quad \text{and} \quad \sigma(O) \leq C\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$ here $C = C(N, \operatorname{diam}(\Omega), T)$ is small enough. Conversely, the existence holds then for compact subset $K \subset\subset \Omega$, one find $C_K > 0$ such that

$$\mu(E \cap (K \times [0,T])) \leq C_K \mathrm{Cap}_{2,1,q'}(E) \ \text{ and } \ \sigma(O \cap K) \leq C_K \mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. In unbounded domain $\mathbb{R}^N \times (0,\infty)$, in [30] asserted that an inequality

$$u_t - \Delta u \geq u^q, u \geq 0 \ \text{ in } \mathbb{R}^N \times (0,\infty), \tag{4.1.8}$$

i. if $q < (N+2)/N$ then the only nonnegative global (in time) solution of above inequality is $u \equiv 0$,

ii. if $q > (N+2)/N$ then there exists global positive solution of above inequality.

More general, see [6], for $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$, (4.1.7) has a nonnegative solution in $\mathbb{R}^N \times (0,\infty)$ (with $A(x,t,\nabla u) = \nabla u$) if and only if

$$\mu(E) \leq C \mathrm{Cap}_{\mathcal{H}_2,q'}(E) \ \text{ and } \ \sigma(O) \leq C \mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{4.1.9}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, here $C = C(N,q)$ is small enough, two capacities $\mathrm{Cap}_{\mathcal{H}_2,q'}, \mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ will be defined in section 2. Note that a necessary and sufficient condition for (4.1.9) holding with $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty)) \backslash \{0\}$ or $\sigma \in \mathfrak{M}^+(\mathbb{R}^N) \backslash \{0\}$ is $q \geq (N+2)/N$. In particular, (4.1.8) has a (global) positive solution if and only if $q \geq (N+2)/N$. It is known that conditions for data μ, σ in problems with absorption are softer than source. Recently, in exponential case, i.e $|u|^{q-1}u$ is replaced by $P(u) \sim \exp(a|u|^q)$, for $a > 0$ and $q \geq 1$ was established in [61].

We consider (4.1.6) and (4.1.7) in Ω_T with Dirichlet boundary conditions when $\operatorname{div}(A(x,t,\nabla u))$ is replaced by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p \in (2 - 1/N, N)$. In [66], showed that for any $q > p-1$, (4.1.6) admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ is diffuse measure i.e absolutely continuous with respect to C_p-parabolic capacity in Ω_T defined on a compact set $K \subset \Omega_T$:

$$C_p(K, \Omega_T) = \inf \left\{ ||\varphi||_X : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega_T) \right\},$$

where $X = \{\varphi : \varphi \in L^p(0,T; W_0^{1,p}(\Omega)), \varphi_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))\}$ endowed with norm $||\varphi||_X = ||\varphi||_{L^p(0,T;W_0^{1,p}(\Omega))} + ||\varphi_t||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$ and χ_K is the characteristic function of K. An improving result was presented in [14] for measures that have good behavior in time, it is based on results of [16] relative to the elliptic case. That is, (4.1.6) has a (renormalized) solution for $q > p-1$ if $\sigma \in L^1(\Omega)$ and $|\mu| \leq f + \omega \otimes F$, where $f \in L^1_+(\Omega_T), F \in L^1_+((0,T))$ and $\omega \in \mathfrak{M}_b^+(\Omega)$ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$ in Ω. Also, (4.1.7) has a (renormalized) nonnegative solution if $\sigma \in L^\infty_+(\Omega)$, $0 \leq \mu \leq \omega \otimes \chi_{(0,T)}$ with $\omega \in \mathfrak{M}_b^+(\Omega)$ and

$$\omega(E) \leq C_1 \mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \ \forall \text{ compact } E \subset \mathbb{R}^N, \quad ||\sigma||_{L^\infty(\Omega)} \leq C_2$$

for some C_1, C_2 small enough. Another improving results are also stated in [15], especially if $q > p-1$, $p > 2$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$ are absolutely continuous with respect to $\text{Cap}_{2,1,q'}$ in Ω_T and $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω then (4.1.6) has a distribution solution.

In [15], we also obtain the existence of solutions for porous medium equation with absorption and data measure : for $q > m > \frac{N-2}{N}$, a sufficient condition for existence solution to the problem

$$u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \quad \text{in} \quad \Omega_T, \quad u = 0 \text{ on } \partial\Omega \times (0,T), \quad \text{and} \quad u(0) = \sigma \quad \text{in} \quad \Omega.$$

is $\mu << \text{Cap}_{2,1,q'}$, $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ if $m \geq 1$ and $\mu << \text{Cap}_{\mathcal{G}_2,\frac{2q}{2(q-1)+N(1-m)}}$, $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ if $\frac{N-2}{N} < m \leq 1$. A necessary condition is $\mu << \text{Cap}_{2,1,\frac{q}{q-\max\{m,1\}}}$ and $\sigma << \text{Cap}_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$. Moreover, if $\mu = \mu_1 \otimes \chi_{[0,T]}$ with $\mu_1 \in \mathfrak{M}_b(\Omega)$ and $\sigma \equiv 0$ then a condition $\mu_1 << \text{Cap}_{\mathbf{G}_2,\frac{q}{q-m}}$ is not only a sufficient but also a necessary for existence of solutions to above problem.

We would like to make a brief survey of quasilinear elliptic equations with absorption, source terms and data measure :

$$-\Delta_p u + |u|^{q-1}u = \omega, \tag{4.1.10}$$
$$-\Delta_p u = |u|^{q-1}u + \omega, u \geq 0 \tag{4.1.11}$$

in Ω with Dirichlet boundary conditions where $1 < p < N$, $q > p-1$. In [16], we proved that the existence solution of equation (4.1.10) holds if $\omega \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to $\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$. Moreover, a necessary condition for existence was also showed in [10, 11]. For problem with source term, it was solved in [68] (also see [69]). Exactly, if $\omega \in \mathfrak{M}_b^+(\Omega)$ has compact support in Ω, then a sufficient and necessary condition for the existence of solutions of problem (4.1.11) is

$$\omega(E) \leq C\text{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \qquad \text{for all compact set } E \subset \Omega$$

where C is a constant only depending on N, p, q and $d(\text{supp}(\omega), \partial\Omega)$. Their construction is based upon sharp estimates of solutions of the problem

$$-\Delta_p u = \omega \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

for nonnegative Radon measures ω in Ω and a deep analysis of the Wolff potential. Corresponding results in case that u^q term is changed by $P(u) \approx \exp(au^\lambda)$ for $a > 0, \lambda > 0$, was given in [16, 62].

In [27], Duzaar and Mingione gave a local pointwise estimate from above of solutions to equation

$$u_t - \text{div}(A(x,t,\nabla u)) = \mu \tag{4.1.12}$$

in Ω_T involving the Wolff parabolic potential $\mathbb{I}_2[|\mu|]$ defined by

$$\mathbb{I}_2[|\mu|](x,t) = \int_0^\infty \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \quad \text{for all} \quad (x,t) \in \mathbb{R}^{N+1},$$

here $\tilde{Q}_\rho(x,t) := B_\rho(x) \times (t-\rho^2/2, t+\rho^2/2)$. Specifically if $u \in L^2(0,T;H^1(\Omega)) \cap C(\Omega_T)$ is a weak solution to above equation with data $\mu \in L^2(\Omega_T)$, then

$$|u(x,t)| \le C \fint_{\tilde{Q}_R(x,t)} |u| dyds + C \int_0^{2R} \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho}, \tag{4.1.13}$$

for any $Q_{2R}(x,t) := B_{2R}(x) \times (t-(2R)^2, t) \subset \Omega_T$, where a constant C only depends on N and the structure of operator A. Moreover, in this paper we show that if $u \ge 0, \mu \ge 0$ we also have local pointwise estimate from below :

$$u(y,s) \ge C^{-1} \sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \tag{4.1.14}$$

for any $Q_r(y,s) \subset \Omega_T$, see section 5, where $r_k = 4^{-k} r$.

From preceding two inequalities, we obtain global pointwise estimates of solution to (4.1.12). For example, if $\mu \in \mathfrak{M}(\mathbb{R}^{N+1})$ with $\mathbb{I}_2[|\mu|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then there exists a distribution solution to (4.1.12) in $\mathbb{R}^{N+1}$ such that

$$-K\mathbb{I}_2[\mu^-](x,t) \le u(x,t) \le K\mathbb{I}_2[\mu^+](x,t) \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1} \tag{4.1.15}$$

and we emphasize that if $u \ge 0, \mu \ge 0$ then

$$u(x,t) \ge K^{-1} \sum_{k=-\infty}^{\infty} \frac{\mu(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1},$$

and for $q > 1$,

$$||u||_{L^q(\mathbb{R}^{N+1})} \approx ||\mathbb{I}_2[\mu]||_{L^q(\mathbb{R}^{N+1})}.$$

Where a constant K only depends on N and the structure of operator A.

Our first aim is to verify that

i. problems (4.1.4) and (4.1.6) have solutions if μ, σ are absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ respectively,

ii. problems (4.1.5) in $\mathbb{R}^{N+1}$ and (4.1.7) in $\mathbb{R}^N \times (0,\infty)$ with data signed measure μ, σ admit a solution if

$$|\mu|(E) \le C\text{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad |\sigma|(O) \le C\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{4.1.16}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. Also, the equation (4.1.7) in a bounded domain Ω_T has a solution if (4.1.16) holds where capacities $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ are exploited instead of $\text{Cap}_{\mathcal{H}_2,q'}, \text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$.

It is worth mention that solutions obtained of (4.1.5) in $\mathbb{R}^{N+1}$ and (4.1.7) in $\mathbb{R}^N \times (0,\infty)$ obey

$$\int_E |u|^q dxdt \le C\text{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1}$$

and we also have an analogous estimate for a solution of (4.1.7) in Ω_T;

$$\int_E |u|^q dxdt \leq C\mathrm{Cap}_{2,1,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1}$$

for some a constant $C > 0$.

In case $\mu \equiv 0$, solutions (4.1.7) in $\mathbb{R}^N \times (0,\infty)$ and Ω_T are accepted the decay estimate

$$-Ct^{-\frac{1}{q-1}} \leq \inf_x u(x,t) \leq \sup_x u(x,t) \leq Ct^{-\frac{1}{q-1}} \text{ for any } \ t > 0.$$

The strategy for establishment above results that is, we rely upon the combination some techniques of quasilinear elliptic equations in two articles [16, 68] with the global pointwise estimate (4.1.15), delicate estimates on Wolff parabolic potential and the stability theorem see [13], Proposition 4.3.17 of this paper. They will be demonstrated in section 6.

We next are interested in global regularity of solutions to quasilinear parabolic equations

$$u_t - \mathrm{div}\,(A(x,t,\nabla u)) = \mu \ \text{ in } \ \Omega_T, \ u = 0 \text{ on } \partial\Omega \times (0,T) \ \text{ and } \ u(0) = \sigma \ \text{ in } \ \Omega. \quad (4.1.17)$$

where domain Ω_T and nonlinearity A are as mentioned at the beginning.

Our aim is to achieve minimal conditions on the boundary of Ω and on nonlinearity A so that the following statement holds

$$|||\nabla u|||_{\mathcal{K}} \leq C||\mathbb{M}_1[\omega]||_{\mathcal{K}}.$$

Here $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$ and $\mathbb{M}_1$ is the first order fractional Maximal parabolic potential defined by

$$\mathbb{M}_1[\omega](x,t) = \sup_{\rho>0} \frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^{N+1}} \ \ \forall\ (x,t) \in \mathbb{R}^{N+1},$$

, a constant C does not depend on u and $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ and $\mathcal{K}$ is a function space. The same question is as above for the elliptic framework studied by N. C. Phuc in [70, 71, 72].

First, we take $\mathcal{K} = L^{p,s}(\Omega_T)$ for $1 \leq p \leq \theta$ and $0 < s \leq \infty$ under a capacity density condition on the domain Ω where $L^{p,s}(\Omega_T)$ is the Lorentz space and a constant $\theta > 2$ depends on the structure of this condition and of nonlinearity A. It follows the recent result in [7], see remark 4.2.18. The capacity density condition is that, the complement of Ω satisfies *uniformly* $2-$*thick*, see section 2. We remark that under this condition, the Sobolev embedding $H_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ for $N > 2$ is valid and it is fulfilled by any domain with Lipschitz boundary, or even of corkscrew type. This condition was used in two papers [70, 72]. Also, it is essentially sharp for higher integrability results, presented in [41, Remark 3.3]. Furthermore, we also assert that if $\frac{\gamma}{\gamma-1} < p < \theta$, $2 \leq \gamma < N+2$, $0 < s \leq \infty$ and $\sigma \equiv 0$ then

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)} \leq C||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}$$

for some a constant C where $L_*^{p,s;(\gamma-1)p}(\Omega_T), L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$ are the Lorentz-Morrey spaces involving "calorie" introduced in section 2. We would like to refer to [55] as the first paper where Lorentz-Morrey estimates for solutions of quasilinear elliptic equations via fractional operators have been obtained.

Next, in order to obtain shaper results, we take $\mathcal{K} = L^{q,s}(\Omega_T, dw)$, the weighted Lorentz spaces with weight in the Muckenhoupht class A_∞ for $q \ge 1$, $0 < s \le \infty$, we require some stricter conditions on the domain Ω and nonlinearity A. A condition on Ω is flat enough in the sense of Reifenberg, essentially, that at boundary point and every scale the boundary of domain is between two hyperplanes at both sides (inside and outside) of domain by a distance which depends on the scale. Conditions on A are that BMO type of A with respect to the $x-$variable is small enough and the derivative of $A(x,t,\zeta)$ with respect to ζ is uniformly bounded. By choosing an appropriate weight we can establish the following important estimates :

a. The Lorentz-Morrey estimates involving "calorie" for $0 < \kappa \le N+2$ is obtained

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\Omega_T)} \le C||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\Omega_T)}.$$

b. Another Lorentz-Morrey estimates is also obtained for $0 < \vartheta \le N$

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)} \le C||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)},$$

where $L_{**}^{q,s;\vartheta}(\Omega_T)$ is introduced in section 2. This estimate implies global Holder-estimate in space variable and L^q-estimate in time, that is for all ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}}||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)} \text{ provided } \ 0 < \vartheta < \min\{q, N\}.$$

In particular, there hold

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}}||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)} + C\rho^{1-\frac{\vartheta}{q}}||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega, L^{q_1}((0,T)))}$$

provided

$$\begin{aligned}
&1 < q_1 \le q < 2,\\
&\max\left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)\right\} < \vartheta \le N.
\end{aligned}$$

Where $L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)$ is the standard Morrey space and

$$||\mu||_{L^{q_2;\vartheta}(\Omega, L^{q_1}((0,T)))} = \sup_{\rho>0, x\in\Omega} \rho^{\frac{\vartheta-N}{q_2}}\left(\int_{B_\rho(x)\cap\Omega}\left(\int_0^T |\mu(y,t)|^{q_1}dt\right)^{\frac{q_2}{q_1}}dy\right)^{\frac{1}{q_2}}.$$

with $q_2 = \frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$. Besides, we also find

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C\rho^{1-\frac{\vartheta}{q}}||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega, L^{q_1}((0,T)))}$$

provided

$$\sigma \equiv 0, \quad q \geq 2, 1 < q_1 \leq q,$$
$$\frac{1}{q-1}\left(2 + q - \frac{2q}{q_1}\right) < \vartheta \leq N.$$

c. A global capacitary estimate is also given

$$\sup_{\substack{\text{compact } K \subset \mathbb{R}^{N+1} \\ \mathrm{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\int_K |\nabla u|^q dxdt}{\mathrm{Cap}_{\mathcal{G}_1,q'}(K)}\right) \leq C \sup_{\substack{\text{compact } K \subset \mathbb{R}^{N+1} \\ \mathrm{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{|\omega|(K)}{\mathrm{Cap}_{\mathcal{G}_1,q'}(K)}\right)^q.$$

To obtain this estimate we employ profound techniques in nonlinear potential theory, see section 4 and Theorem 4.2.22.

We utilize some ideas (in the quasilinear elliptic framework) in articles of N.C. Phuc [70, 72, 71] during we establish above estimates.

We would like to emphasize that above estimates is also true for solutions to equation (4.1.17) in $\mathbb{R}^{N+1}$ with data μ (of course still true for (4.1.17) in $\mathbb{R}^N \times (0, \infty)$) with data μ provided $\mathbb{I}_2[|\mu|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$ see Theorem 4.2.25 and 4.2.27. Moreover, a global pointwise estimates of gradient of solutions is obtained when A is independent of space variable x, that is

$$|\nabla u(x,t)| \leq C\mathbb{I}_1[|\mu|](x,t) \quad \text{a.e } (x,t) \in \mathbb{R}^{N+1}$$

see Theorem 4.2.5.

Our final aim is to obtain existence results for the quasilinear Riccati type parabolic problems (4.1.1) where $B(x,t,u,\nabla u) = |\nabla u|^q$ for $q > 1$. The strategy we use in order to prove these existence results is that using Schauder Fixed Point Theorem and all above estimates and the stability Theorem see [13], Proposition 4.3.17 in section 3. They will be carried out in section 9. By our methods in the paper, we can treat general equations (4.1.1), where

$$|B(x,t,u,\nabla u)| \leq C_1|u|^{q_1} + C_2|\nabla u|^{q_2}, \; q_1, q_2 > 1,$$

with constant coefficients $C_1, C_2 > 0$.

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4.2 Main Results

Throughout the paper, we assume that Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$. Besides, we always denote $\Omega_T = \Omega \times (0,T)$, $T_0 = \mathrm{diam}(\Omega) + T^{1/2}$ and $Q_\rho(x,t) =$

$B_\rho(x) \times (t-\rho^2, t)$ $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t-\rho^2/2, t+\rho^2/2)$ for $(x,t) \in \mathbb{R}^{N+1}$ and $\rho > 0$. We always assume that $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory vector valued function, i.e. A is measurable in (x,t) and continuous with respect to ∇u for each fixed (x,t) and satisfies (4.1.2) and (4.1.3). This article is divided into three parts. First part, we study the existence problems for the quasilinear parabolic equations with absorption and source terms

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + |u|^{q-1}u = \mu \text{ in } \Omega_T, \\ u = 0 \quad \text{ on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \Omega, \end{cases} \tag{4.2.1}$$

and

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \text{ in } \Omega_T, \\ u = 0 \quad \text{ on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \Omega, \end{cases} \tag{4.2.2}$$

where $q > 1$, and μ, σ are Radon measures.

In order to state our results, let us introduce some definitions and notations. If D is either a bounded domain or whole $\mathbb{R}^l$ for $l \in \mathbb{N}$, we denote by $\mathfrak{M}(D)$ (resp. $\mathfrak{M}_b(D)$) the set of Radon measure (resp. bounded Radon measures) in D. Their positive cones are $\mathfrak{M}^+(D)$ and $\mathfrak{M}_b^+(D)$ respectively. For $R \in (0,\infty]$, we define the $R-$truncated Riesz parabolic potential $\mathbb{I}_\alpha$ and Fractional Maximal parabolic potential $\mathbb{M}_\alpha$, $\alpha \in (0, N+2)$, on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{I}_\alpha^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \frac{d\rho}{\rho} \text{ and } \mathbb{M}_\alpha^R[\mu](x,t) = \sup_{0<\rho<R} \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \tag{4.2.3}$$

for all (x,t) in $\mathbb{R}^{N+1}$. If $R = \infty$, we drop it in expressions of (4.2.3).
We denote by $\mathcal{H}_\alpha$ the Heat kernel of order $\alpha \in (0, N+2)$:

$$\mathcal{H}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \quad \text{in} \quad \mathbb{R}^{N+1},$$

and $\mathcal{G}_\alpha$ the parabolic Bessel kernel of order $\alpha > 0$:

$$\mathcal{G}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-t-\frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \quad \text{in} \quad \mathbb{R}^{N+1},$$

see [4], where $C_\alpha = \left((4\pi)^{N/2}\Gamma(\alpha/2)\right)^{-1}$. It is known that $\mathcal{F}(\mathcal{H}_\alpha)(x,t) = (|x|^2 + it)^{-\alpha/2}$ and $\mathcal{F}(\mathcal{G}_\alpha)(x,t) = (1+|x|^2+it)^{-\alpha/2}$. We define the parabolic Riesz potential $\mathcal{H}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{H}_\alpha[\mu](x,t) = (\mathcal{H}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{H}_\alpha(x-y,t-s)d\mu(y,s) \text{ for any } \quad (x,t) \text{ in } \mathbb{R}^{N+1},$$

the parabolic Bessel potential $\mathcal{G}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{G}_\alpha[\mu](x,t) = (\mathcal{G}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{G}_\alpha(x-y,t-s)d\mu(y,s) \text{ for any } (x,t) \text{ in } \mathbb{R}^{N+1}.$$

We also define $\mathbf{I}_\alpha, \mathbf{G}_\alpha, 0<\alpha<N$ the Riesz, Bessel potential of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ by

$$\mathbf{I}_\alpha[\mu](x)=\int_0^\infty \frac{\mu(B_\rho(x))}{\rho^{N-\alpha}}\frac{d\rho}{\rho} \text{ and } \mathbf{G}_\alpha[\mu](x)=\int_{\mathbb{R}^N}\mathbf{G}_\alpha(x-y)d\mu(y) \text{ for any } x \text{ in } \mathbb{R}^N,$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α, see [2].
Several different capacities will be used over the paper. For $1<p<\infty$, the $(\mathcal{H}_\alpha,p)$-capacity, $(\mathcal{G}_\alpha,p)$-capacity of a Borel set $E\subset\mathbb{R}^{N+1}$ are defined by

$$\text{Cap}_{\mathcal{H}_\alpha,p}(E)=\inf\left\{\int_{\mathbb{R}^{N+1}}|f|^p dxdt : f\in L^p_+(\mathbb{R}^{N+1}), \mathcal{H}_\alpha * f\geq \chi_E\right\} \text{ and}$$
$$\text{Cap}_{\mathcal{G}_\alpha,p}(E)=\inf\left\{\int_{\mathbb{R}^{N+1}}|f|^p dxdt : f\in L^p_+(\mathbb{R}^{N+1}), \mathcal{G}_\alpha * f\geq \chi_E\right\}.$$

The $W^{2,1}_p-$capacity of compact set $E\subset\mathbb{R}^{N+1}$ is defined by

$$\text{Cap}_{2,1,p}(E)=\inf\left\{||\varphi||^p_{W^{2,1}_p(\mathbb{R}^{N+1})} : \varphi\in S(\mathbb{R}^{N+1}), \varphi\geq 1 \text{ in a neighborhood of } E\right\},$$

where

$$||\varphi||_{W^{2,1}_p(\mathbb{R}^{N+1})}=||\varphi||_{L^p(\mathbb{R}^{N+1})}+||\frac{\partial\varphi}{\partial t}||_{L^p(\mathbb{R}^{N+1})}+||\nabla\varphi||_{L^p(\mathbb{R}^{N+1})}+\sum_{i,j=1,2,...,N}||\frac{\partial^2\varphi}{\partial x_i\partial x_j}||_{L^p(\mathbb{R}^{N+1})}.$$

We remark that thanks to Richard J. Bagby's result (see [4]) we obtain the equivalent of capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{\mathcal{G}_2,p}$, i.e, for any compact set $K\subset\mathbb{R}^{N+1}$ there holds

$$C^{-1}\text{Cap}_{2,1,p}(K)\leq \text{Cap}_{\mathcal{G}_2,p}(K)\leq C\text{Cap}_{2,1,p}(K)$$

for some $C=C(N,p)$, see Corollary (4.4.18) in section 4.
The $(\mathbf{I}_\alpha,p)$-capacity, $(\mathbf{G}_\alpha,p)$-capacity of a Borel set $O\subset\mathbb{R}^N$ are defined by

$$\text{Cap}_{\mathbf{I}_\alpha,p}(O)=\inf\left\{\int_{\mathbb{R}^N}|g|^p dx : g\in L^p_+(\mathbb{R}^N), \mathbf{I}_\alpha * g\geq \chi_O\right\} \text{ and}$$
$$\text{Cap}_{\mathbf{G}_\alpha,p}(O)=\inf\left\{\int_{\mathbb{R}^N}|g|^p dx : g\in L^p_+(\mathbb{R}^N), \mathbf{G}_\alpha * g\geq \chi_O\right\}.$$

In our first three Theorems, we present global pointwise potential estimates for solutions to quasilinear parabolic problems

$$\begin{cases} u_t-div\,(A(x,t,\nabla u))=\mu \text{ in } \Omega_T,\\ u=0 \quad \text{ on } \partial\Omega\times(0,T),\\ u(0)=\sigma \quad \text{in } \Omega,\end{cases} \tag{4.2.4}$$

and

$$\begin{cases} u_t-div\,(A(x,t,\nabla u))=\mu \text{ in } \mathbb{R}^N\times(0,\infty),\\ u(0)=\sigma \quad \text{in } \mathbb{R}^N,\end{cases} \tag{4.2.5}$$

and

$$u_t-div\,(A(x,t,\nabla u))=\mu \text{ in } \mathbb{R}^{N+1}. \tag{4.2.6}$$

Theorem 4.2.1 *There exists a constant K depending on N, Λ_1, Λ_2 such that for any $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ there is a distribution solution u of* (4.2.4) *which satisfies*

$$-K\mathbb{I}_2^{2T_0}[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \le u \le K\mathbb{I}_2^{2T_0}[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \text{ in } \Omega_T \tag{4.2.7}$$

Remark 4.2.2 *Since* $\sup_{x\in\mathbb{R}^N} \mathbb{I}_\alpha[\sigma^\pm \otimes \delta_{\{t=0\}}](x,t) \le \frac{\sigma^\pm(\Omega)}{(N+2-\alpha)(2|t|)^{\frac{N+2-\alpha}{2}}}$ *for any $t \neq 0$ with $0 < \alpha < N+2$. Thus, if $\mu \equiv 0$, then we obtain the decay estimate :*

$$-\frac{K\sigma^-(\Omega)}{N(2t)^{\frac{N}{2}}} \le \inf_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u(x,t) \le \frac{K\sigma^+(\Omega)}{N(2t)^{\frac{N}{2}}} \text{ for any } 0 < t < T.$$

Theorem 4.2.3 *There exists a constant C depending on N, Λ_1, Λ_2 such that for any $\mu \in \mathfrak{M}_b^+(\Omega_T), \sigma \in \mathfrak{M}_b^+(\Omega)$, there is a distribution solution u of* (4.2.4) *satisfying for a.e $(y,s) \in \Omega_T$ and $B_r(y) \subset \Omega$*

$$u(y,s) \ge C\sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N} + C\sum_{k=0}^{\infty} \frac{(\sigma \otimes \delta_{\{t=0\}})(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N} \tag{4.2.8}$$

where $r_k = 4^{-k}r$.

Remark 4.2.4 *The Theorem 4.2.3 is also true when we replace the assumption* (4.1.3) *by a weaker one*

$$\langle A(x,t,\zeta), \zeta\rangle \ge \Lambda_2|\zeta|^2, \quad \langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda\rangle > 0$$

for every $(\lambda, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N$, $\lambda \neq \zeta$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Theorem 4.2.5 *Let K be the constant in Theorem 4.2.1. Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that $I_2[|\omega|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$. Then, there is a distribution solution u to* (4.2.6) *with data $\mu = \omega$ satisfying*

$$-K\mathbb{I}_2[\omega^-] \le u \le K\mathbb{I}_2[\omega^+] \text{ in } \mathbb{R}^{N+1} \tag{4.2.9}$$

such that the following statements hold.

a. *If $\omega \ge 0$, there exists $C_1 = C_1(N, \Lambda_1, \Lambda_2)$ such that for a.e $(x,t) \in \mathbb{R}^{N+1}$*

$$u(x,t) \ge C_1 \sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \tag{4.2.10}$$

In particular, for any $q > \frac{N+2}{N}$

$$C_2^{-1}||\mathcal{H}_2[\omega]||_{L^q(\mathbb{R}^{N+1})} \le ||u||_{L^q(\mathbb{R}^{N+1})} \le C_2||\mathcal{H}_2[\omega]||_{L^q(\mathbb{R}^{N+1})}. \tag{4.2.11}$$

with $C_2 = C_2(N, \Lambda_1, \Lambda_2)$.

b. *If A is independent of space variable x and satisfies* (4.2.27)*, then there exists $C_2 = C_2(N, \Lambda_1, \Lambda_2)$ such that*

$$|\nabla u| \le C_2\mathbb{I}_1[|\omega|] \text{ in } \mathbb{R}^{N+1}. \tag{4.2.12}$$

c. *If* $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ *with* $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ *and* $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, *then* $u = 0$ *in* $\mathbb{R}^N \times (-\infty, 0)$ *and* $u|_{\mathbb{R}^N \times [0,\infty)}$ *is a distribution solution to* (4.2.5).

Remark 4.2.6 *For* $q > \frac{N+2}{N}$, *we alway have the following claim :*

$$||\mathcal{H}_2[\mu + \omega \otimes \delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \approx ||\mathcal{H}_2[\mu]||_{L^q(\mathbb{R}^{N+1})} + ||\mathbf{I}_{2/q}[\sigma]||_{L^q(\mathbb{R}^{N+1})}$$

for every $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ *and* $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$.

Remark 4.2.7 *For* $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ *if* $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ *for some* $(x_0,t_0) \in \mathbb{R}^{N+1}$ *then for any* $0 < \beta \leq \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{loc}(\mathbb{R}^{N+1})$ *for any* $0 < s < \frac{N+2}{N+2-\beta}$. *However, for* $0 < \beta < \alpha < N+2$, *one can find* $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ *such that* $\mathbb{I}_\alpha[\omega] \equiv \infty$ *and* $\mathbb{I}_\beta[\omega] < \infty$ *in* $\mathbb{R}^{N+1}$, *see Appendix section.*

The next four theorems provide the existence of solutions to quasilinear parabolic equations with absorption and source terms. For convenience, we always denote by q' the conjugate exponent of $q \in (1,\infty)$ i.e $q' = \frac{q}{q-1}$.

Theorem 4.2.8 *Let* $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ *and* $\sigma \in \mathfrak{M}_b(\Omega)$. *Suppose that* μ, σ *are absolutely continuous with respect to the capacities* $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ *in* Ω_T, Ω *respectively. Then there exists a distribution solution* u *of* (4.2.1) *satisfying*

$$-K\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \leq u \leq K\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \quad in \ \Omega_T.$$

Here the constant K *is in Theorem 4.2.1.*

Theorem 4.2.9 *Let* K *be the constant in Theorem 4.2.1. Let* $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ *and* $\sigma \in \mathfrak{M}_b(\Omega)$. *There exists a constant* $C_1 = C_1(N, q, \Lambda_1, \Lambda_2, diam(\Omega), T)$ *such that if*

$$|\mu|(E) \leq C_1 Cap_{2,1,q'}(E) \quad and \quad |\sigma|(O) \leq C_1 Cap_{\mathbf{G}_{\frac{2}{q}},q'}(O). \tag{4.2.13}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, *then the problem* (4.2.2) *has a distribution solution* u *satisfying*

$$-\frac{Kq}{q-1}\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \leq u \leq \frac{Kq}{q-1}\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \quad in \ \ \Omega_T. \tag{4.2.14}$$

Besides, for every compact set $E \subset \mathbb{R}^{N+1}$ *there holds*

$$\int_E |u|^q dxdt \leq C_2 Cap_{2,1,q'}(E) \tag{4.2.15}$$

where $C_2 = C_2(N, q, \Lambda_1, \Lambda_2, T_0)$.

Remark 4.2.10 *From* (4.2.15) *we get if* $q > \frac{N+2}{N}$,

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \leq C\rho^{N+2-2q'} \quad for\ any \ \ \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1},$$

if $q = \frac{N+2}{N}$,

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \leq C\left(\log(1/\rho)\right)^{-\frac{1}{q-1}} \quad \textit{for any } \ \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}, 0<\rho<1/2$$

for some $C = C(N, q, \Lambda_1, \Lambda_2, T_0)$, *see Remark 4.4.14.*

Remark 4.2.11 *In the sub-critical case* $1 < q < \frac{N+2}{N}$, *since the capacity* $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ *of a single are positive thus the conditions* (4.2.13) *hold for some constant* $C_1 > 0$ *provided* $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$. *Moreover, in the super-critical case* $q > \frac{N+2}{N}$, *we have*

$$Cap_{2,1,q'}(E) \geq c_1|E|^{1-\frac{2q'}{N+2}} \quad \textit{and} \quad Cap_{\mathbf{G}_{\frac{2}{q}},q'}(O) \geq c_2|O|^{1-\frac{2}{(q-1)N}}$$

for every Borel sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, *thus if* $\mu \in L^{\frac{N+2}{2q'},\infty}(\Omega_T)$ *and* $\sigma \in L^{\frac{(q-1)N}{2},\infty}(\Omega)$ *then* (4.2.13) *holds for some constant* $C_1 > 0$. *In addition, if* $\mu \equiv 0$, *then* (4.2.14) *implies for any* $0 < t < T$,

$$-c_3(T_0)t^{-\frac{1}{q-1}} \leq \inf_{x\in\Omega} u(x,t) \leq \sup_{x\in\Omega} u(x,t) \leq c_3(T_0)t^{-\frac{1}{q-1}},$$

since $|\sigma|(B_\rho(x)) \leq c_4(T_0)\rho^{N-\frac{2}{q-1}}$ *for all* $x \in \mathbb{R}^N$, $0 < \rho < 2T_0$.

Theorem 4.2.12 *Let* K *be the constant in Theorem 4.2.1 and* $q > 1$. *If* $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ *is absolutely continuous with respect to the capacity* $Cap_{2,1,q'}$ *in* $\mathbb{R}^{N+1}$, *then there exists a distribution solution* $u \in L^\gamma_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ *for any* $1 \leq \gamma < \frac{2q}{q+1}$ *to problem*

$$u_t - div\left(A(x,t,\nabla u)\right) + |u|^{q-1}u = \omega \ \textit{in} \ \mathbb{R}^{N+1}, \tag{4.2.16}$$

which satisfies

$$-K\mathbb{I}_2[\omega^-] \leq u \leq K\mathbb{I}_2[\omega^+] \quad \textit{in} \ \mathbb{R}^{N+1}. \tag{4.2.17}$$

Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ *with* $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ *then* $u = 0$ *in* $\mathbb{R}^N \times (-\infty, 0)$ *and* $u|_{\mathbb{R}^N\times[0,\infty)}$ *is a distribution solution to problem*

$$\begin{cases} u_t - div\left(A(x,t,\nabla u)\right) + |u|^{q-1}u = \mu \ \textit{in} \ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad \textit{in} \ \ \mathbb{R}^N. \end{cases} \tag{4.2.18}$$

Remark 4.2.13 *The measure* $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ *is absolutely continuous with respect to the capacity* $Cap_{2,1,q'}$ *in* $\mathbb{R}^{N+1}$ *if and only if* μ, σ *are absolutely continuous with respect to the capacities* $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ *in* $\mathbb{R}^{N+1}, \mathbb{R}^N$ *respectively.*

Existence result of the problem (4.2.2) on $\mathbb{R}^{N+1}$ or on $\mathbb{R}^N \times (0,\infty)$ is similar to Theorem 4.2.9 presented in the following Theorem, where the capacities $\text{Cap}_{\mathcal{H}_2,q'}, \text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ are used in place of respectively $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$.

Theorem 4.2.14 *Let K be the constant in Theorem 4.2.1 and $q > \frac{N+2}{N}$, $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists a constant $C_1 = C_1(N, q, \Lambda_1, \Lambda_2)$ such that if*

$$|\omega|(E) \leq C_1 \, Cap_{\mathcal{H}_{2,q'}}(E) \tag{4.2.19}$$

for every compact set $E \subset \mathbb{R}^{N+1}$, then the problem

$$u_t - div\,(A(x,t,\nabla u)) = |u|^{q-1}u + \omega \ \ in\ \mathbb{R}^{N+1} \tag{4.2.20}$$

has a distribution solution $u \in L^{\gamma}_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ for any $1 \leq \gamma < \frac{2q}{q+1}$ satisfying

$$-\frac{Kq}{q-1}\mathbb{I}_2[\omega^-] \leq u \leq \frac{Kq}{q-1}\mathbb{I}_2[\omega^+] \quad in \quad \mathbb{R}^{N+1}. \tag{4.2.21}$$

Moreover, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to problem

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \ \ in\ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad in\ \ \mathbb{R}^N. \end{cases} \tag{4.2.22}$$

In addition, for any compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$\int_E |u|^q dxdt \leq C_2 \, Cap_{\mathcal{H}_{2,q'}}(E) \tag{4.2.23}$$

for some $C_2 = C_2(N, q, \Lambda_1, \Lambda_2)$.

Remark 4.2.15 *The measure $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ satisfies (4.2.19) if and only if*

$$|\mu|(E) \leq C \, Cap_{\mathcal{H}_{2,q'}}(E) \ \ and \ \ |\sigma|(O) \leq C \, Cap_{\mathbf{I}_{\frac{2}{q}},q'}(O).$$

for every compact sets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$, where $C = C_3 C_1$, $C_3 = C_3(N, q)$.

Remark 4.2.16 *If $\omega \in L^{\frac{N+2}{2q'},\infty}(\mathbb{R}^{N+1})$ then (4.2.19) holds for some constant $C_1 > 0$. Moreover, if $\omega = \sigma \otimes \delta_{\{t=0\}}$ with $\sigma \in \mathfrak{M}_b(\mathbb{R}^N)$, then from (4.2.21) we get the decay estimate :*

$$-c_1 t^{-\frac{1}{q-1}} \leq \inf_{x \in \mathbb{R}^N} u(x,t) \leq \sup_{x \in \mathbb{R}^N} u(x,t) \leq c_1 t^{-\frac{1}{q-1}} \ for\ any \ \ t > 0,$$

since $|\sigma|(B_\rho(x)) \leq c_2 \rho^{N - \frac{2}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

Second part, we establish global regularity in weighted-Lorentz and Lorentz-Morrey on gradient of solutions to problem (4.2.4). For this purpose, we need a capacity density condition imposed on Ω. That is, the complement of Ω satisfies *uniformly p-thick with constants* c_0, r_0, i.e, for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^N \backslash \Omega$ there holds

$$\text{Cap}_p(\overline{B_r(x)} \cap (\mathbb{R}^N \backslash \Omega), B_{2r}(x)) \geq c_0 \text{Cap}_p(\overline{B_r(x)}, B_{2r}(x)) \tag{4.2.24}$$

where the involved capacity of a compact set $K \subset B_{2r}(x)$ is given as follows

$$\mathrm{Cap}_p(K, B_{2r}(x)) = \inf\left\{\int_{B_{2r}(x)} |\nabla\phi|^p dy : \phi \in C_c^\infty(B_{2r}(x)), \phi \geq \chi_K\right\}. \tag{4.2.25}$$

In order to obtain better regularity we need a stricter condition on Ω which is expressed in the following way. We say that Ω is a $(\delta, R_0)-$Reifenberg flat domain for $\delta \in (0,1)$ and $R_0 > 0$if for every $x_0 \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, ..., z_n\}$, which may depend on r and x_0, so that in this coordinate system $x_0 = 0$ and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}. \tag{4.2.26}$$

We remark that this class of flat domains is rather wide since it includes C^1, Lipschitz domains with sufficiently small Lipschitz constants and fractal domains. Besides, it has many important roles in the theory of minimal surfaces and free boundary problems, this class was first appeared in a work of Reifenberg (see [74]) in the context of a Plateau problem. Its properties can be found in [37, 38, 78].

On the other hand, it is well-known that in general, conditions (4.1.2) and (4.1.3) on the nonlinearity $A(x,t,\zeta)$ are not enough to ensure higher integral of gradient of solutions to problem (4.2.4), we need to assume that A satisfies

$$\langle A_\zeta(x,t,\zeta)\xi, \xi\rangle \geq \Lambda_2|\xi|^2, \quad |A_\zeta(x,t,\zeta)| \leq \Lambda_1 \tag{4.2.27}$$

for every $(\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{(0,0)\}$ and a.e $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where Λ_1, Λ_2 are constants in (4.1.2) and (4.1.3). We also require that the nonlinearity A satisfies a smallness condition of BMO type in the x-variable. We say that $A(x,t,\zeta)$ satisfies a (δ, R_0)-BMO condition for some $\delta, R_0 > 0$ with exponent $s > 0$ if

$$[A]_s^{R_0} := \sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R}, 0<r\leq R_0} \left(\fint_{Q_r(y,s)} (\Theta(A, B_r(y))(x,t))^s \, dxdt\right)^{\frac{1}{s}} \leq \delta,$$

where

$$\Theta(A, B_r(y))(x,t) := \sup_{\zeta\in\mathbb{R}^N\backslash\{0\}} \frac{|A(x,t,\zeta) - \overline{A}_{B_r(y)}(t,\zeta)|}{|\zeta|}$$

and $\overline{A}_{B_r(y)}(t,\zeta)$ is denoted the average of $A(t,.,\zeta)$ over the cylinder $B_r(y)$, i.e,

$$\overline{A}_{B_r(y)}(t,\zeta) := \fint_{B_r(y)} A(x,t,\zeta)dx = \frac{1}{|B_r(y)|}\int_{B_r(y)} A(x,t,\zeta)dx.$$

The above condition was appeared in [21]. It is easy to see that the $(\delta, R_0)-$BMO condition on A is satisfied when A is continuous or has small jump discontinuities with respect to x.

In this paper, $\mathbb{M}$ denotes the Hardy-Littlewood maximal function defined for each locally integrable function f in $\mathbb{R}^{N+1}$ by

$$\mathbb{M}(f)(x,t) = \sup_{\rho>0} \fint_{\tilde{Q}_\rho(x,t)} |f(y,s)|dyds \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

We verify that $\mathbb{M}$ is bounded operator from $L^1(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^s(\mathbb{R}^{N+1})$ ($L^{s,\infty}(\mathbb{R}^{N+1})$) to itself for $s > 1$, see [76, 77].

We recall that a positive function $w \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ is called an A_∞ if there are two positive constants C and ν such that

$$w(E) \le C \left(\frac{|E|}{|Q|} \right)^\nu w(Q)$$

for all cylinder $Q = \tilde{Q}_\rho(x,t)$ and all measurable subsets E of Q. The pair (C,ν) is called the A_∞ constant of w and is denoted by $[w]_{A_\infty}$.

For a weight function $w \in A_\infty$, the weighted Lorentz spaces $L^{q,s}(D,dw)$ with $0 < q < \infty$, $0 < s \le \infty$ and a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L^{q,s}(D,dw)} := \begin{cases} \left(q \int_0^\infty \left(\rho^q w\left(\{(x,t) \in D : |g(x,t)| > \rho\}\right)\right)^{\frac{s}{q}} \frac{d\rho}{\rho} \right)^{1/s} < \infty & \text{if } s < \infty, \\ \sup_{\rho>0} \rho w\left(\{(x,t) \in D : |g(x,t)| > \rho\}\right)^{1/q} < \infty & \text{if } s = \infty. \end{cases}$$

Here we write $w(E) = \int_E w(x,t)dxdt$ for a measurable set $E \subset \mathbb{R}^{N+1}$. Obviously, $||g||_{L^{q,q}(D,dw)} = ||g||_{L^q(D,dw)}$, thus we have $L^{q,q}(D,dw) = L^q(D,dw)$. As usual, when $w \equiv 1$ we simply write $L^{q,s}(D)$ instead of $L^{q,s}(D,dw)$.
We now state the next results of the paper.

Theorem 4.2.17 *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of (4.2.4) with data μ and σ such that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0 then for any $1 \le p < \theta$ and $0 < s \le \infty$,*

$$||\mathbb{M}(|\nabla u|)||_{L^{p,s}(\Omega_T)} \le C_1 ||\mathbb{M}_1[\omega]||_{L^{p,s}(Q)}. \tag{4.2.28}$$

Here $\theta = \theta(N, \Lambda_1, \Lambda_1, c_0) > 2$ and $C_1 = C_1(N, \Lambda_1, \Lambda_2, p, s, c_0, T_0/r_0)$ and $Q = B_{diam(\Omega)}(x_0) \times (0,T)$ which $\Omega \subset B_{diam(\Omega)}(x_0)$.
Especially, when $1 < p < 2$, then

$$||\mathbb{M}(|\nabla u|)||_{L^p(\Omega_T)} \le C_2 \left(||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} + ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)} \right), \tag{4.2.29}$$

where $C_2 = C_2(N, \Lambda_1, \Lambda_2, p, c_0, T_0/r_0)$.

Remark 4.2.18 *If $\frac{N+2}{N+1} < p < 2$, there hold*

$$||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} \le C_1 ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} \quad \text{and} \quad ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)} \le C_1 ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)}$$

for some $C_1 = C_1(N,p)$. From (4.2.29) we obtain

$$|||\nabla u|||_{L^p(\Omega_T)} \le C_2 ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} + C_2 ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)} \quad \text{provided } \frac{N+2}{N+1} < p < 2.$$

We should mention that if $\sigma \equiv 0$, then

$$||\mathbb{M}_1[\omega]||_{L^{p,s}(\mathbb{R}^{N+1})} \le C_2 ||\mu||_{L^{\frac{q(N+2)}{N+2+q},s}(\Omega_T)}.$$

and we get [7, Theorem 1.2] from estimate (4.2.28).

In order to state the next results, we need to introduce Lorentz-Morrey spaces $L_*^{q,s;\theta}(D)$ involving "calorie" with a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L_*^{q,s;\kappa}(D)} := \sup_{0<\rho<\text{diam}(D),(x,t)\in D} \rho^{\frac{\kappa-N-2}{q}} ||g||_{L^{q,s}(\tilde{Q}_\rho(x,t)\cap D)} < \infty,$$

where $0 < \kappa \leq N+2$, $0 < q < \infty$, $0 < s \leq \infty$. Clearly, $L_*^{q,s;N+2}(D) = L^{q,s}(D)$. Moreover, when $q = s$ the space $L_*^{q,s;\theta}(D)$ will be denoted by $L_*^{q;\theta}(D)$.
The following theorem provides an estimate on gradient in Lorentz-Morrey spaces.

Theorem 4.2.19 *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of* (4.2.4) *with data μ and σ such that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0 then for any $1 \leq p < \theta$ and $0 < s \leq \infty$, $2 - \gamma_0 < \gamma < N+2$, $\gamma \leq \frac{N+2}{p} + 1$,*

$$\begin{aligned} ||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;p(\gamma-1)}(\Omega_T)} &\leq C_1 ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)} \\ &+ C_2 \sup_{0<R\leq T_0,(y_0,s_0)\in\Omega_T} \left(R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \right). \end{aligned} \quad (4.2.30)$$

Here θ is in Theorem 4.2.17, $\gamma_0 = \gamma_0(N, \Lambda_1, \Lambda_1, c_0) \in (0, 1/2]$ and $C_1 = C_1(N, \Lambda_1, \Lambda_2, p, s, \gamma, c_0, T_0/r_0)$, $C_2 = C_2(N, \Lambda_1, \Lambda_2, p, s, \gamma, c_0)$. Besides, if $\frac{\gamma}{\gamma-1} < p < \theta$, $2 - \gamma_0 < \gamma < N+2$, $0 < s \leq \infty$ and $\mu \in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$, $\sigma \equiv 0$, then u is a unique renormalized solution satisfied

$$||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)} \leq C_3 ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}, \quad (4.2.31)$$

where $C_3 = C_3(N, \Lambda_1, \Lambda_2, p, s, \gamma, c_0, T_0/r_0)$.

Theorem 4.2.20 *Suppose that A satisfies* (4.2.27). *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of* (4.2.4) *with data μ, σ such that the following holds. For any $w \in A_\infty$, $1 \leq q < \infty$, $0 < s \leq \infty$ we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, [w]_{A_\infty}) \in (0, 1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \leq \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L^{q,s}(\Omega_T,dw)} \leq C ||\mathbb{M}_1[\omega]||_{L^{q,s}(\Omega_T,dw)}. \quad (4.2.32)$$

Here C depends on $N, \Lambda_1, \Lambda_2, q, s, [w]_{A_\infty}$ and T_0/R_0.

Next results are actually consequences of Theorem 4.2.20. For our purpose, we introduce another Lorentz-Morrey spaces $L_{**}^{q,s;\theta}(O_1 \times O_2)$, is the set of measurable functions g on $O_1 \times O_2$ such that

$$||g||_{L_{**}^{q,s;\vartheta}(O_1\times O_2)} := \sup_{0<\rho<\text{diam}(O_1),x\in O_1} \rho^{\frac{\vartheta-N}{q}} ||g||_{L^{q,s}((B_\rho(x)\cap O_1)\times O_2))} < \infty,$$

where O_1, O_2 are Borel sets in $\mathbb{R}^N$ and $\mathbb{R}$ respectively, $0 < \vartheta \le N$, $0 < q < \infty$, $0 < s \le \infty$. Obviously, $L_{**}^{q,s;N}(D) = L^{q,s}(D)$. For simplicity of notation, we write $L_{**}^{q;\vartheta}(D)$ instead of $L_{**}^{q,s;\vartheta}(D)$ when $q = s$. Moreover,

$$||g||_{L_{**}^{q,q;\vartheta}(O_1\times O_2)} = ||G||_{L^{q;\vartheta}(O_1)},$$

where $G(x) = ||g(x,.)||_{L^q(O_1)}$ and $L^{q;\vartheta}(O_1)$ is the usual Morrey space, i.e the spaces of all measurable functions f on O_1 with

$$||f||_{L^{q;\vartheta}(O_1)} := \sup_{0<\rho<\mathrm{diam}(O_1), y\in O_1} \rho^{\frac{\vartheta-N}{q}} ||f||_{L^q(B_\rho(y)\cap O_1)} < \infty.$$

Theorem 4.2.21 *Suppose that A satisfies (4.2.27). Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. Let s_0 be in Theorem 4.2.20. There exists a distribution solution of (4.2.4) with data μ, σ such that the following holds.*

a. *For any $1 \le q < \infty$, $0 < s \le \infty$ and $0 < \kappa \le N+2$ we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \kappa) \in (0,1)$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_*^{q,s;\kappa}(\Omega_T)} \le C_1 ||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\Omega_T)}. \tag{4.2.33}$$

Here C_1 depends on $N, \Lambda_1, \Lambda_2, q, s, \kappa$ and T_0/R_0.

b. *For any $1 \le q < \infty$, $0 < s \le \infty$ and $0 < \vartheta \le N$ we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q, s, \vartheta) \in (0,1)$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)} \le C_2 ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)}. \tag{4.2.34}$$

for some $C_2 = C_2(N, \Lambda_1, \Lambda_2, q, s, \vartheta, T_0/R_0)$. Especially, when $q = s$ and $0 < \vartheta < \min\{N, q\}$, there holds for any ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |osc_{B_\rho\cap\overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \le C_3 \rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)}. \tag{4.2.35}$$

for some $C_3 = C_3(N, \Lambda_1, \Lambda_2, q, \vartheta, T_0/R_0)$.

The following global capacitary estimates on gradient.

Theorem 4.2.22 *Suppose that A satisfies (4.2.27). Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. Let s_0 be in Theorem 4.2.20. There exists a distribution solution of (4.2.4) with data μ, σ such that following holds. For any $1 < q < \infty$, we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ such that if Ω is a (δ, R_0)- Reifenberg flat domain and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then*

$$\sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\int_{K\cap\Omega_T} |\nabla u|^q dxdt}{Cap_{\mathcal{G}_1,q'}(K)}\right) \le C_1 \sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\omega(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right)^q, \tag{4.2.36}$$

and if $q > \frac{N+2}{N+1}$,

$$\sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{H}_1,q'}(K)>0}}\left(\frac{\int_{K\cap\Omega_T}|\nabla u|^q dxdt}{Cap_{\mathcal{H}_1,q'}(K)}\right)\leq C_2\sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{H}_1,q'}(K)>0}}\left(\frac{\omega(K)}{Cap_{\mathcal{H}_1,q'}(K)}\right)^q. \tag{4.2.37}$$

Where $C_1 = C_1(N,\Lambda_1,\Lambda_2,q,T_0/R_0,T_0)$ *and* $C_2 = C_2(N,\Lambda_1,\Lambda_2,q,T_0/R_0)$.

Remark 4.2.23 *We have if* $1 < q < 2$, *then*

$$C^{-1}\sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{G}_1,q'}(K)>0}}\left(\frac{(|\sigma|\otimes\delta_{\{t=0\}})(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right)\leq\sup_{\substack{compact\ O\subset\mathbb{R}^{N}\\ Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)>0}}\left(\frac{|\sigma|(O)}{Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)}\right)$$
$$\leq C\sup_{\substack{compact\ K\subset\mathbb{R}^{N+1}\\ Cap_{\mathcal{G}_1,q'}(K)>0}}\left(\frac{(|\sigma|\otimes\delta_{\{t=0\}})(K)}{Cap_{\mathcal{G}_1,q'}(K)}\right)$$

for $C = C(N,q)$, *if* $\frac{N+2}{N+1} < q < 2$, *then above estimate is true when two capacities* $Cap_{\mathcal{G}_1,q'}$, ,$Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}$ *are replaced by* $Cap_{\mathcal{H}_1,q'}$,$Cap_{\mathbf{I}_{\frac{2}{q}-1},q'}$ *respectively, see Remark 4.4.34.*

Remark 4.2.24 *Above results also hold when* $[A]_s^{R_0}$ *is replaced by* $\{A\}_s^{R_0}$:

$$\{A\}_s^{R_0} := \sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R},0<r\leq R_0}\left(\fint_{Q_r(y,s)}\left(\Theta(A,Q_r(y,s))(x,t)\right)^s dxdt\right)^{\frac{1}{s}}\leq\delta$$

where

$$\Theta(A,Q_r(y,s))(x,t) := \sup_{\zeta\in\mathbb{R}^N\setminus\{0\}}\frac{|A(x,t,\zeta)-\overline{A}_{Q_r(y,s)}(\zeta)|}{|\zeta|}$$

and $\overline{A}_{Q_r(y,s)}(\zeta)$ *is denoted the average of* $A(.,.,\zeta)$ *over the cylinder* $Q_r(y,s)$, *i.e,*

$$\overline{A}_{Q_r(y,s)}(\zeta) := \fint_{Q_r(y,s)}A(x,t,\zeta)dxdt = \frac{1}{|Q_r(y,s)|}\int_{Q_r(y,s)}A(x,t,\zeta)dxdt.$$

Next results are corresponding estimates of gradient for domain $\mathbb{R}^N\times(0,\infty)$ or whole $\mathbb{R}^{N+1}$.

Theorem 4.2.25 *Let* $\theta\in(2,N+2)$ *be in Theorem 4.2.17 and* $\omega\in\mathfrak{M}(\mathbb{R}^{N+1})$. *There exists a distribution solution u of* (4.2.6) *with data* $\mu=\omega$ *such that the following statements hold*

a. *For any* $\frac{N+2}{N+1} < p < \theta$ *and* $0 < s \leq \infty$,

$$|||\nabla u|||_{L^{p,s}(\mathbb{R}^{N+1})}\leq C_1||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}, \tag{4.2.38}$$

for some $C_1 = C_1(N,\Lambda_1,\Lambda_2,p,s)$.

b. *For any* $\frac{N+2}{N+1} < p < \theta$ *and* $0 < s \leq \infty$, $2-\gamma_0 < \gamma < N+2$ *and* $\gamma \leq \frac{N+2}{p}+1$,

$$\begin{aligned}|||\nabla u|||_{L_*^{p,s;p(\gamma-1)}(\mathbb{R}^{N+1})} &\leq C_2||\mathbb{M}_\gamma[|\omega|]||_{L^\infty(\mathbb{R}^{N+1})}\\ &+C_2\sup_{R>0,(y_0,s_0)\in\mathbb{R}^{N+1}}\left(R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\omega|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\right),\end{aligned} \tag{4.2.39}$$

provided $\mathbb{I}_2[|\omega|](x_0,t_0) < \infty$ *for some* $(x_0,t_0) \in \mathbb{R}^{N+1}$.
Also, if $\omega \in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})$ *with* $p > \frac{\gamma}{\gamma-1}$ *then*

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\mathbb{R}^{N+1})} \leq C_3||\omega||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})}, \tag{4.2.40}$$

for some $\gamma_0 = \gamma_0(N,\Lambda_1,\Lambda_2) \in (0,\frac{1}{2}]$ *and* $C_i = C_i(N,\Lambda_1,\Lambda_2,p,s,\gamma)$, $i = 2,3$.

c. *The statement* **c** *in Theorem 4.2.5 is true.*

Remark 4.2.26 *Let* $s > 1$. *For* $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ *implies* $\mathbb{I}_2[|\omega|] < \infty$ *a.e in* $\mathbb{R}^{N+1}$ *if and only if* $s \leq N+2$.

Theorem 4.2.27 *Suppose that* A *satisfies* (4.2.27). *Let* s_0 *be in Theorem 4.2.20. Let* $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ *with* $\mathbb{I}_2[|\omega|](x_0,t_0) < \infty$ *for some* $(x_0,t_0) \in \mathbb{R}^{N+1}$. *There exists a distribution solution of* (4.2.6) *with data* $\mu = \omega$ *such that following statements hold,*

a. *For any* $w \in A_\infty$, $1 \leq q < \infty$, $0 < s \leq \infty$ *we find* $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}) \in (0,1)$ *such that if* $[A]_{s_0}^\infty \leq \delta$ *then*

$$|||\nabla u|||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \leq C_1||\mathbb{M}_1[|\omega|]||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \tag{4.2.41}$$

Here C_1 *depends on* $N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}$.

b. *For any* $\frac{N+2}{N+1} < q < \infty$, $0 < s \leq \infty$ *and* $0 < \kappa \leq N+2$ *we find* $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,\kappa) \in (0,1)$ *such that if* $[A]_{s_0}^\infty \leq \delta$ *then*

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})} \leq C_2||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})}. \tag{4.2.42}$$

Here C_2 *depends on* $N,\Lambda_1,\Lambda_2,q,s,\kappa$.

c. *For any* $\frac{N+2}{N+1} < q < \infty$, $0 < s \leq \infty$ *and* $0 < \vartheta \leq N$ *one find* $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,\vartheta) \in (0,1)$ *such that if* $[A]_{s_0}^\infty \leq \delta$ *then*

$$|||\nabla u|||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})} \leq C_3||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})}. \tag{4.2.43}$$

Here C_3 *depends on* $N,\Lambda_1,\Lambda_2,q,s,\vartheta$. *Especially, when* $q = s$ *and* $0 < \vartheta < \min\{N,q\}$, *there holds for any ball* $B_\rho \subset \mathbb{R}^N$

$$\left(\int_{\mathbb{R}}|osc_{B_\rho}u(t)|^q dt\right)^{\frac{1}{q}} \leq C_4\rho^{1-\frac{\vartheta}{q}}||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})}. \tag{4.2.44}$$

for some $C_4 = C_4(N,\Lambda_1,\Lambda_2,q,\vartheta)$.

d. *For any* $\frac{N+2}{N+1} < q < \infty$*, one find* $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ *such that if* $[\mathcal{A}]_{s_0}^{\infty} \leq \delta$ *then*

$$\sup_{\substack{\text{compact } K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{\int_K |\nabla u|^q dxdt}{Cap_{\mathcal{H}_1,q'}(K)}\right) \leq C_5 \sup_{\substack{\text{compact } K \subset \mathbb{R}^{N+1} \\ Cap_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{|\omega|(K)}{Cap_{\mathcal{H}_1,q'}(K)}\right)^q, \quad (4.2.45)$$

for some $C_5 = C_5(N, \Lambda_1, \Lambda_2, q)$.

e. *The statement* **c** *in Theorem 4.2.5 is true.*

The following some estimates for norms of $\mathbb{M}_1[\omega]$ in $L_*^{q;\kappa}(\mathbb{R}^{N+1})$ and $L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})$

Proposition 4.2.28 *Let* $1 < \kappa \leq N+2$, $0 < \vartheta \leq N$ *and* $q, q_1 > 1$. *Suppose that* $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. *Then* $\mathbb{M}_1[\mu] \leq 2^{N+2}\mathbb{I}_1[\mu]$ *and*

a. *If* $q > \frac{\kappa}{\kappa-1}$ *then*

$$||\mathbb{I}_1[\mu]||_{L_*^{q;\kappa}(\mathbb{R}^{N+1})} \leq C_1 ||\mu||_{L_*^{\frac{q\kappa}{q+\kappa};\kappa}(\mathbb{R}^{N+1})}. \quad (4.2.46)$$

Here C_1 *depends on* N, q, κ.

b. *If* $1 < q < 2$ *then*

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \leq \mathbf{I}_{\frac{2}{q}-1}[\mu_1](x) \quad (4.2.47)$$

where μ_1 *is a nonnegative radon measure in* $\mathbb{R}^N$ *defined by* $\mu_1(A) = \mu(A \times \mathbb{R})$ *for every Borel set* $A \subset \mathbb{R}^N$. *In particular,*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq ||\mathbf{I}_{\frac{2}{q}-1}[\mu_1]||_{L^{q;\vartheta}(\mathbb{R}^N)} \quad (4.2.48)$$

and if $\vartheta > \frac{2-q}{q-1}$ *there holds*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq C_2 ||\mu_1||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} \quad (4.2.49)$$

for some $C_2 = C_2(N, q, \vartheta)$.

c. *If* $\frac{2q}{q+2} < q_1 \leq q$ *then*

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \leq \mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2](x) \quad (4.2.50)$$

where $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})}dx$. *In particular,*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq ||\mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2]||_{L^{q;\vartheta}(\mathbb{R}^N)} \quad (4.2.51)$$

and if $\vartheta > \frac{1}{q-1}\left(2 + q - \frac{2q}{q_1}\right)$ *there holds*

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \leq C_3 ||\mu_2||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N)} = C_3 ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N, L^{q_1}(\mathbb{R}))} \quad (4.2.52)$$

for some $C_3 = C_3(N, q, \vartheta)$.

The proof of Proposition 4.2.28 will performed at the end of section 8.

Remark 4.2.29 *Let* $1 < q < 2$, $0 < \vartheta \le N$ *and* $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. *From* (4.2.48) *and* (4.2.49) *in Proposition 4.2.28 we assert that*

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}}]||_{L^{q;\vartheta}_{**}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}-1}[|\sigma|]||_{L^{q;\vartheta}(\mathbb{R}^N)},$$

and

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}}]||_{L^{q;\vartheta}_{**}(\mathbb{R}^{N+1})} \le C_1||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} \quad \text{if } \vartheta > \frac{2-q}{q-1},$$

for some $C_1 = C_1(N, q, \vartheta)$.
Furthermore, from preceding inequality and (4.2.52) *in Proposition 4.2.28 we can state that*

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|]||_{L^{q;\vartheta}_{**}(\mathbb{R}^{N+1})} \le C_2||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} + C_2||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))},$$

provided

$$\begin{aligned} &1 < q_1 \le q < 2, \\ &\max\left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)\right\} < \vartheta \le N, \end{aligned}$$

for some $C_2 = C_2(N, q, \vartheta)$. *Where*

$$||\mu||_{L^{q_2;\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))} = \sup_{\rho>0,x\in\mathbb{R}^N} \rho^{\frac{\vartheta-N}{q_2}} \left(\int_{B_\rho(x)} \left(\int_{\mathbb{R}} |\mu(y,t)|^{q_1} dt\right)^{\frac{q_2}{q_1}} dy\right)^{\frac{1}{q_2}},$$

with $q_2 = \frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$.

Final part, we prove the existence solutions for the quasilinear Riccati type parabolic problems

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = |\nabla u|^q + \mu \quad \text{in } \Omega_T, \\ u = 0 \quad\quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad\ \text{in } \ \Omega, \end{cases} \tag{4.2.53}$$

and

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |\nabla u|^q + \mu \text{ in } \ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad \text{in } \ \mathbb{R}^N, \end{cases} \tag{4.2.54}$$

and

$$u_t - div\,(A(x,t,\nabla u)) = |\nabla u|^q + \mu \text{ in } \ \mathbb{R}^{N+1}, \tag{4.2.55}$$

where $q > 1$.

The following result is considered in subcritical case this means $1 < q < \frac{N+2}{N+1}$, to obtain existence solutions in this case we need data μ, σ to be finite measures and small enough.

Theorem 4.2.30 *Let* $1 < q < \frac{N+2}{N+1}$ *and* $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$. *There exists* $\varepsilon_0 = \varepsilon_0(N, \Lambda_1, \Lambda_2, q) > 0$ *such that if*

$$|\Omega_T|^{-1+\frac{q'}{N+2}} \left(|\mu|(\Omega_T) + |\omega|(\Omega)\right) \le \varepsilon_0,$$

the problem (4.2.53) *has a distribution solution* u, *satisfied*

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le C \left(|\mu|(\Omega_T) + |\omega|(\Omega)\right)$$

for some $C = C(N, \Lambda_1, \Lambda_2, q) > 0$.

In the next results are concerned in critical and supercritical case.

Theorem 4.2.31 *Suppose that* $\mathbb{R}^N \backslash \Omega$ *satisfies uniformly* $2-$*thick with constants* c_0, r_0. *Let* θ *be in Theorem 4.2.17,* $q \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$, $\mu \in \mathfrak{M}_b(\Omega_T)$ *and* $\sigma \in \mathfrak{M}_b(\Omega)$. *Assume that* $\sigma \equiv 0$ *when* $q \ge \frac{N+4}{N+2}$. *There exists* $\varepsilon_0 = \varepsilon_0(N, \Lambda_1, \Lambda_2, q, c_0, T_0/r_0) > 0$ *such that if*

$$||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \le \varepsilon_0,$$

then the problem (4.2.53) *has a distribution solution* u *satisfying*

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\Omega_T)} \le C||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + C||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \tag{4.2.56}$$

for some $C = C(N, \Lambda_1, \Lambda_2, q, c_0, T_0/r_0)$.

We remark that a necessary condition for existence $\sigma \in \mathfrak{M}_b(\Omega)\backslash\{0\}$ with $\mathbb{M}_1[|\sigma| \otimes \delta_{\{t=0\}}] \in L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})$ is $\frac{N+2}{N+1} \le q < \frac{N+4}{N+2}$.

Theorem 4.2.32 *Suppose that* A *satisfies* (4.2.27). *Let* s_0 *be the constant in Theorem 4.2.20. Let* $q \ge \frac{N+2}{N+1}$ *and* $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$, *set* $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. *There exists* $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0, 1)$ *such that* Ω *is* (δ, R_0)*-Reifenberg flat domain* Ω *and* $[A]_{s_0}^{R_0} \le \delta$ *for some* R_0 *and the following holds. The problem* (4.2.53) *has a distribution solution* u *if one of the following three cases is true :*

Case a. *A is a linear operator and*

$$\omega(K) \le C_1 Cap_{\mathcal{G}_1, q'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1} \tag{4.2.57}$$

with a constant C_1 *small enough.*

Case b. *there holds*

$$\omega(K) \le C_2 Cap_{\mathcal{G}_1, (q+\varepsilon)'}(K) \quad \text{for every compact subset } K \subset \mathbb{R}^{N+1} \tag{4.2.58}$$

where $\varepsilon > 0$ *and* C_2 *is a constant small enough.*

Case c. $\begin{cases} q > \frac{N+2}{N+1}, \\ q \ge \frac{N+4}{N+2} \quad \text{if } \sigma \equiv 0, \\ ||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}, ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \\ \quad \text{is small enough.} \end{cases}$

*A solution u corresponds to **Case a**, **b** and **c** satisfying*

$$\int_K |\nabla u|^q dxdt \le C_3 C_1^q Cap_{\mathcal{G}_1,q'}(K) \quad \textit{for every compact subset } K \subset \mathbb{R}^{N+1},$$

$$\int_K |\nabla u|^{q+\varepsilon} dxdt \le C_4 C_2^{q+\varepsilon} Cap_{\mathcal{G}_1,(q+\varepsilon)'}(K) \quad \textit{for every compact subset } K \subset \mathbb{R}^{N+1},$$

and

$$|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \le C_5 ||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + C_5 ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)},$$

respectively. Where C_3, C_4, C_5 are constants depended on $N, \Lambda_1, \Lambda_2, q, \varepsilon, T_0/R_0$, besides C_3, C_4 also depend on T_0.

Since $\text{Cap}_{\mathcal{G}_1,s}(B_r(0) \times \{t = 0\}) = 0$ for all $r > 0$ and $0 < s \le 2$, see Remark 4.4.13 thus if there is $\sigma \in \mathfrak{M}_b(\Omega)\backslash\{0\}$ satisfying $(|\sigma| \otimes \delta_{\{t=0\}})(E) \le \text{Cap}_{\mathcal{G}_1,s}(E)$ for every compact subsets $E \subset \mathbb{R}^{N+1}$ then we must have $s > 2$.

The above results are not sharp in the case A is a nonlinear operator. However, if A is Holder continuous with respect to x we can prove that problem (4.2.53) has a distribution solution with data having compact support in Ω_T.

Theorem 4.2.33 *Let Ω be a bounded open subset in $\mathbb{R}^N$ such that the boundary of Ω is in $C^{1,\beta}$ with $\beta \in (0, 1)$. Suppose that A satisfies* (4.2.27) *and*

$$|A(x, t, \zeta) - A(y, t, \zeta)| \le \Lambda_3 |x - y|^\beta |\zeta| \tag{4.2.59}$$

for every $x, y \in \Omega$ and $t > 0, \zeta \in \mathbb{R}^N$. Let $\Omega' \subset\subset \Omega$ and set $d = dist(\Omega', \Omega) > 0$. Then, there exist $C = C(N, q, \Lambda_1, \Lambda_2, \Lambda_3, \beta, d, \Omega, T) > 0$ and $\Lambda = \Lambda(N, q, \Lambda_1, \Lambda_2, \Lambda_3, \beta, d, \Omega, T) > 0$ such that for any $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$ with $supp(\mu) \subset \Omega' \times [0, T]$, $supp(\sigma) \subset \Omega'$, the problem (4.2.53) *has a distribution solution u, satisfying*

$$|\nabla u(x, t)| \le \Lambda \mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x, t) \quad a.e\ (x, t) \in \Omega_T \tag{4.2.60}$$

provided that one of the following two cases is true :

Case a. $1 < q < 2$ *and*

$$|\mu|(E) \le C\, Cap_{\mathcal{G}_1,q'}(E) \text{ and } |\sigma|(O) \le C\, Cap_{\mathbf{G}_{\frac{2}{q}-1},q'}(O) \tag{4.2.61}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$.

Case b. $q \ge 2$ *and* $\sigma \equiv 0$,

$$|\mu|(E) \le C\, Cap_{\mathcal{G}_1,q'}(E) \tag{4.2.62}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$.

Remark 4.2.34 *If $q > \frac{N+2}{N+1}$, $\mu \equiv 0$ and **Case a** is satisfied then* (4.2.60) *gives the decay estimate :*

$$\sup_{x\in\Omega} |\nabla u(x,t)| \leq c_1 t^{-\frac{1}{2(q-1)}} \ \forall\ 0 < t < T,$$

since $|\sigma|(B_\rho(x)) \leq c_2(T_0)\rho^{N-\frac{2-q}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

We have an **important** Proposition.

Proposition 4.2.35 *All the existence results considered the bounded domain Ω_T have recently been presented in above Theorems, if $\sigma \in L^1(\Omega)$ then the solutions obtained in those Theorems are renormalized solutions.*

Theorem 4.2.36 *Let $\theta \in (2, N+2)$ be in Theorem 4.2.17, $q \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists $C_1 = C_1(N, \Lambda_1, \Lambda_2, q) > 0$ such that if*

$$||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq C_1$$

then the problem (4.2.55) *has a distribution solution $u \in L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$ such that*

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\mathbb{R}^{N+1})} \leq C_2||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \tag{4.2.63}$$

for some $C_2 = C_2(N, \Lambda_1, \Lambda_2, q)$. Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to problem (4.2.54).

Theorem 4.2.37 *Suppose that A satisfies* (4.2.27)*. Let $q > \frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$. Let s_0 be the constant in Theorem 4.2.20, δ in Theorem 4.2.32. There exists $C_1 = C_1(N, \Lambda_1, \Lambda_2, q) > 0$ such that if $[A]^\infty_{s_0} \leq \delta$ and*

$$||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq C_1 \tag{4.2.64}$$

then the problem (4.2.55) *has a distribution solution u satisfying* (4.2.63)*. Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to problem* (4.2.54).

From Remark 4.2.26, we see that if $q \leq 2$ then (4.2.64) follows the assumption $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$.

When A is independent of space variable, we can improve the result of Theorem 4.2.37 as follows :

Theorem 4.2.38 *Suppose that A is independent of space variable and satisfies* (4.2.27)*. Let $q > \frac{N+2}{N+1}$ and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. Assume that $\mathbb{I}_2[|\omega|](x_0, t_0) < \infty$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$. There exist constants $\Lambda = \Lambda(N, \Lambda_1, \Lambda_2, q)$ and $C = C(N, \Lambda_1, \Lambda_2, q)$ such that the problem*

$$u_t - div\,(A(t, \nabla u)) = |\nabla u|^q + \omega \ \textit{in}\ \ \mathbb{R}^{N+1} \tag{4.2.65}$$

has a distribution solution u, satisfying

$$|\nabla u| \leq \Lambda \mathbb{I}_1[\omega] \quad in \ \ \mathbb{R}^{N+1}, \tag{4.2.66}$$

provided that for all compact subset $E \subset \mathbb{R}^{N+1}$

$$|\omega|(E) \leq C\, Cap_{\mathcal{H}_1,q'}(E). \tag{4.2.67}$$

Furthermore, when $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ *with* $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ *and* $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ *then* $u = 0$ *in* $\mathbb{R}^N \times (-\infty, 0)$ *and* $u|_{\mathbb{R}^N \times [0,\infty)}$ *is a distribution solution to problem*

$$\begin{cases} u_t - div\,(A(t,\nabla u)) = |\nabla u|^q + \mu \ in \ \ \mathbb{R}^N \times (0,\infty), \\ u(0) = \sigma \quad in \quad \mathbb{R}^N. \end{cases} \tag{4.2.68}$$

Remark 4.2.39 *If* $\frac{N+2}{N+1} < q < 2$, $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ *satisfies* (4.2.67) *if and only if*

$$|\mu|(E) \leq C' C\, Cap_{\mathcal{H}_1,q'}(E) \ \ and \ |\sigma|(O) \leq C' C\, Cap_{\mathbf{I}_{\frac{2}{q}-1},q'}(O) \tag{4.2.69}$$

for all compact subsets $E \subset \mathbb{R}^{N+1}$ *and* $O \subset \mathbb{R}^N$, *where* $C' = C'(N,q)$.

Remark 4.2.40 *If* $\omega = \sigma \otimes \delta_{\{t=0\}}$ *then* (4.2.66) *follows the decay estimate :*

$$\sup_{x \in \mathbb{R}^N} |\nabla u(x,t)| \leq c_1 t^{-\frac{1}{2(q-1)}} \ \forall\, 0 < t < T,$$

since $|\sigma|(B_\rho(x)) \leq c_2 \rho^{N - \frac{2-q}{q-1}}$ *for any* $B_\rho(x) \subset \mathbb{R}^N$.

4.3 The notion of solutions and some properties

Although the notion of renormalized solutions becomes more and more familiar in the theory of quasilinear parabolic equations with measure data, it is still necessary to present below some main aspects concerning this notion. Let Ω be a bounded domain in $\mathbb{R}^N$, $(a,b) \subset\subset \mathbb{R}$. If $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega \times (a,b))$ the space of measures in $\Omega \times (a,b)$ which are absolutely continuous with respect to the C_2-capacity defined on a compact set $K \subset \Omega \times (a,b)$ by

$$C_2(K, \Omega \times (a,b)) = \inf\left\{||\varphi||_W : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega \times (a,b))\right\}. \tag{4.3.1}$$

where $W = \{z : z \in L^2(a,b,H_0^1(\Omega)), z_t \in L^2(a,b,H^{-1}(\Omega))\}$ endowed with norm $||\varphi||_W = ||\varphi||_{L^2(a,b,H_0^1(\Omega))} + ||\varphi_t||_{L^2(a,b,H^{-1}(\Omega))}$ and χ_K is the characteristic function of K.

We also denote $\mathfrak{M}_s(\Omega \times (a,b))$ the space of measures in $\Omega \times (a,b)$ with support on a set of zero C_2-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b(\Omega \times (a,b))$ and $\mu_s \in \mathfrak{M}_s(\Omega \times (a,b))$. We recall that any $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b(\Omega \times (a,b))$ can be decomposed under the form $\mu_0 = f - \mathrm{div} g + h_t$ where $f \in L^1(\Omega \times (a,b))$, $g \in L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $h \in L^2(a,b,H_0^1(\Omega))$

and (f,g,h) is said to be decomposition of μ_0. Set $\widehat{\mu_0} = \mu_0 - h_t = f - \text{div}g$. In the general case $\widehat{\mu_0} \notin \mathfrak{M}(\Omega\times(a,b))$, but we write, for convenience,

$$\int_{\Omega\times(a,b)} w d\widehat{\mu_0} := \int_{\Omega\times(a,b)} (fw + g.\nabla w)dxdt, \qquad \forall w \in L^2(a,b,H_0^1(\Omega))\cap L^\infty(\Omega\times(a,b)).$$

However, for $\sigma \in \mathfrak{M}_b(\Omega)$ and $t_0 \in (a,b)$ then $\sigma\otimes\delta_{\{t=t_0\}} \in \mathfrak{M}_0(\Omega\times(a,b))$ if and only if $\sigma \in L^1(\Omega)$, see [26]. We also have that for $\sigma \in \mathfrak{M}_b(\Omega)$, $\sigma\otimes\chi_{[a,b]} \in \mathfrak{M}_0(\Omega\times(a,b))$ if and only if σ is absolutely continuous with respect to the $\text{Cap}_{\mathbf{G}_1,2}$-capacity, see [26].

For $k>0$ and $s\in\mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\},-k\}$. We recall that if u is a measurable function defined and finite a.e. in $\Omega\times(a,b)$, such that $T_k(u) \in L^2(a,b,H_0^1(\Omega))$ for any $k>0$, there exists a measurable function $v : \Omega\times(a,b) \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\le k}v$ a.e. in $\Omega\times(a,b)$ and for all $k>0$. We define the gradient ∇u of u by $v = \nabla u$.
We recall the definition of a renormalized solution given in [65].

Definition 4.3.1 *Suppose that $B \in C(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega\times(a,b))$ and $\sigma \in L^1(\Omega)$. A measurable function u is a **renormalized solution** of*

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = B(u,\nabla u) + \mu \ \ in\ \Omega\times(a,b), \\ u = 0 \qquad on\ \ \partial\Omega\times(a,b), \\ u(a) = \sigma \quad in\ \ \Omega, \end{cases} \tag{4.3.2}$$

if there exists a decomposition (f,g,h) of μ_0 such that

$$\begin{aligned} v = u - h \in L^s(a,b,W_0^{1,s}(\Omega)) \cap L^\infty(a,b,L^1(\Omega))\ \forall s \in \left[1,\frac{N+2}{N+1}\right) \\ T_k(v) \in L^2(a,b,H_0^1(\Omega))\ \forall k>0, B(u,\nabla u) \in L^1(\Omega\times(a,b)) \end{aligned} \tag{4.3.3}$$

and :

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0)=0$,

$$\begin{aligned} &-\int_\Omega S(\sigma)\varphi(a)dx - \int_{\Omega\times(a,b)} \varphi_t S(v)dxdt + \int_{\Omega\times(a,b)} S'(v)A(x,t,\nabla u)\nabla\varphi dxdt \\ &+\int_{\Omega\times(a,b)} S''(v)\varphi A(x,t,\nabla u).\nabla v dxdt = \int_{\Omega\times(a,b)} S'(v)\varphi B(u,\nabla u)dxdt + \int_{\Omega\times(a,b)} S'(v)\varphi d\widehat{\mu_0}, \end{aligned} \tag{4.3.4}$$

for any $\varphi \in L^2(a,b,H_0^1(\Omega))\cap L^\infty(\Omega\times(a,b))$ such that $\varphi_t \in L^2(a,b,H^{-1}(\Omega)) + L^1(\Omega\times(a,b))$ and $\varphi(.,b)=0$;

(ii) for any $\phi \in C(\overline{\Omega}\times[a,b])$,

$$\lim_{m\to\infty}\frac{1}{m}\int_{\{m\le v<2m\}} \phi A(x,t,\nabla u)\nabla v dxdt = \int_{\Omega\times(a,b)} \phi d\mu_s^+ \quad and \tag{4.3.5}$$

$$\lim_{m\to\infty}\frac{1}{m}\int_{\{-m\ge v>-2m\}} \phi A(x,t,\nabla u)\nabla v dxdt = \int_{\Omega\times(a,b)} \phi d\mu_s^- . \tag{4.3.6}$$

Remark 4.3.2 *If $\mu \in L^1(\Omega \times (a,b))$, then we have the following estimates :*

$$||u||_{L^{\frac{N+2}{N},\infty}(\Omega\times(a,b))} \le C_1 \left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega \times (a,b))\right) \quad and$$
$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega\times(a,b))} \le C_1 \left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega \times (a,b))\right),$$

where $C_1 = C_1(N, \Lambda_1, \Lambda_2)$, see [13, Remark 4.9].
In particular,

$$||u||_{L^1(\Omega\times(a,b))} \le C_2(diam(\Omega) + (b-a)^{1/2})^2 \left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega \times (a,b))\right) \quad and$$
$$|||\nabla u|||_{L^1(\Omega\times(a,b))} \le C_2(diam(\Omega) + (b-a)^{1/2}) \left(||\sigma||_{L^1(\Omega)} + |\mu|(\Omega \times (a,b))\right),$$

where $C_2 = C_2(N, \Lambda_1, \Lambda_2)$.

Remark 4.3.3 *It is easy to see that u is a weak solution of problem (4.3.2) in $\Omega \times (a,b)$ with $\mu \in L^2(\Omega \times (a,b))$, $\sigma \in H_0^1(\Omega)$ and $B \equiv 0$ then $U = \chi_{[a,b]}u$ is a unique renormalized solution of*

$$\begin{cases} U_t - div\,(A(x,t,\nabla U)) = \chi_{(a,b)}\mu + (\chi_{[a,b)}\sigma)_t & in\ \Omega \times (c,b), \\ U = 0 & on\ \partial\Omega \times (c,b), \\ U(c) = 0 & in\ \Omega, \end{cases}$$

for any $c < a$.

Remark 4.3.4 *Let $\Omega' \subset\subset \Omega$ and $a < a' < b' < b$. For a nonnegative function $\eta \in C_c^\infty(\Omega' \times (a',b'))$, from (4.3.4) we have*

$$(\eta S(v))_t - \eta_t S(v) + S'(v)A(x,t,\nabla u)\nabla\eta - div\left(S'(v)\eta A(x,t,\nabla u)\right)$$
$$+ S''(v)\eta A(x,t,\nabla u)\nabla v = S'(v)\eta f + \nabla\left(S'(v)\eta\right).g - div\left(S'(v)\eta g\right)$$

in $\mathcal{D}'(\Omega' \times (a',b'))$ Thus, $(\eta S(v))_t \in L^2(a',b',H^{-1}(\Omega')) + L^1(D)$ and we have the following estimate

$$\begin{aligned} ||\left(\eta S(v)\right)_t||_{L^2(a',b',H^{-1}(\Omega'))+L^1(D)} &\le C||S||_{W^{2,\infty}(\mathbb{R})} \big(||\eta_t v||_{L^1(D)} \\ &+ |||\nabla u||\nabla\eta|||_{L^1(D)} + ||\eta|\nabla u|\chi_{|v|\le M}||_{L^2(D)} + ||\eta|\nabla u||\nabla v|\chi_{|v|\le M}||_{L^2(D)} \\ &+ ||\eta f||_{L^1(D)} + ||\eta|\nabla u|^2\chi_{|v|\le M|}||_{L^1(D)} + ||\eta|g|^2||_{L^1(D)} + ||\eta|g|||_{L^2(D)}\big) \end{aligned} \tag{4.3.7}$$

with $D = \Omega' \times (a',b')$ and $supp(S') \subset [-M,M]$.

We recall the following important results, see [13].

Proposition 4.3.5 *Let $\{\mu_n\}$ be a bounded in $\mathfrak{M}_b(\Omega \times (a,b))$ and σ_n be a bounded in $L^1(\Omega)$. Let u_n be a renormalized solution of (4.2.4) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$ and initial data σ_n. If $\{f_n\}$ is bounded in $L^1(\Omega_T)$, $\{g_n\}$ bounded in $L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $\{h_n\}$ convergent in $L^2(a,b,H_0^1(\Omega))$, then, up to a subsequence, $\{u_n\}$ converges to a function u in $L^1(\Omega \times (a,b))$. Moreover, if $\{\mu_n\}$ is a bounded in $L^1(\Omega\times(a,b))$ then $\{u_n\}$ is convergent in $L^s(a,b,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$.*

We say that a sequence of bounded measures $\{\mu_n\}$ in $\Omega \times (a,b)$ converges to a bounded measure μ in $\Omega \times (a,b)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty} \int_{\Omega\times(a,b)} \varphi d\mu_n = \int_{\Omega\times(a,b)} \varphi d\mu \quad \text{for all } \varphi \in C(\Omega\times(a,b)) \cap L^\infty(\Omega\times(a,b))).$$

We recall the following fundamental stability result of [13].

Theorem 4.3.6 *Suppose that $B \equiv 0$. Let $\sigma \in L^1(\Omega)$ and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(\Omega\times(a,b)),$$

with $f \in L^1(\Omega\times(a,b)), g \in L^2(\Omega\times(a,b),\mathbb{R}^N)$, $h \in L^2(a,b,H_0^1(\Omega))$ and $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega\times(a,b))$. Let $\sigma_n \in L^1(\Omega)$ and

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathfrak{M}_b(\Omega\times(a,b))$$

with $f_n \in L^1(\Omega\times(a,b)), g_n \in L^2(\Omega\times(a,b),\mathbb{R}^N), h_n \in L^2(a,b,H_0^1(\Omega))$, and $\rho_n, \eta_n \in \mathfrak{M}_b^+(\Omega\times(a,b))$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega\times(a,b)), \rho_n^2, \eta_n^2 \in L^2(\Omega\times(a,b),\mathbb{R}^N)$ and $\rho_{n,s}, \eta_{n,s} \in \mathfrak{M}_s^+(\Omega\times(a,b))$. Assume that $\{\mu_n\}$ is a bounded in $\mathfrak{M}_b(\Omega\times(a,b))$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ converge to σ, f, g, h in $L^1(\Omega)$,weakly in $L^1(\Omega\times(a,b))$,in $L^2(\Omega\times(a,b),\mathbb{R}^N)$,in $L^2(a,b,H_0^1(\Omega))$ respectively and $\{\rho_n\}, \{\eta_n\}$ converge to μ_s^+, μ_s^- in the narrow topology of measures ; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(\Omega\times(a,b))$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $L^2(\Omega\times(a,b),\mathbb{R}^N)$.
Let $\{u_n\}$ be a sequence of renormalized solutions of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n \text{ in } \Omega\times(a,b), \\ u_n = 0 \qquad \text{on } \partial\Omega\times(a,b), \\ u_n(a) = \sigma_n \quad \text{in } \Omega, \end{cases} \tag{4.3.8}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$. Then up to a subsequence, $\{u_n\}$ converges a.e. in $\Omega\times(a,b)$ to a renormalized solution u of (4.3.2), and $\{v_n\}$ converges a.e. in $\Omega\times(a,b)$ to $v = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla v_n\}$ converge respectively to $\nabla u, \nabla v$ a.e in $\Omega\times(a,b)$, and $\{T_k(v_n)\}$ converges to $T_k(v)$ strongly in $L^2(a,b,H_0^1(\Omega))$ for any $k > 0$.

In order to apply above Theorem, we need some the following properties concerning approximate measures of $\mu \in \mathfrak{M}_b^+(\Omega\times(a,b))$, see [13].

Proposition 4.3.7 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b^+(\Omega\times(a,b))$ with $\mu_0 \in \mathfrak{M}_0(\Omega\times(a,b)) \cap \mathfrak{M}_b^+(\Omega\times(a,b))$ and $\mu_s \in \mathfrak{M}_s^+(\Omega\times(a,b))$. Let $\{\varphi_n\}$ be sequence of standard mollifiers in $\mathbb{R}^{N+1}$. Then, there exist a decomposition (f,g,h) of μ_0 and $f_n, g_n, h_n \in C_c^\infty(\Omega\times(a,b))$, $\mu_{n,s} \in C_c^\infty(\Omega\times(a,b)) \cap \mathfrak{M}_b^+(\Omega\times(a,b))$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega\times(a,b)), L^2(\Omega\times(a,b),\mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$, $\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s}, \mu_{n,s}$ converge to μ, μ_s in the narrow topology respectively, $0 \le \mu_n \le \varphi_n * \mu$ and*

$$||f_n||_{L^1(\Omega\times(a,b))} + \|g_n\|_{L^2(\Omega\times(a,b),\mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega\times(a,b)) \le 2\mu(\Omega\times(a,b)).$$

Proposition 4.3.8 *Let $\mu = \mu_0 + \mu_s, \mu_n = \mu_{n,0} + \mu_{n,s} \in \mathfrak{M}_b^+(\Omega \times (a,b))$ with $\mu_0, \mu_{n,0} \in \mathfrak{M}_0(\Omega\times(a,b))\cap\mathfrak{M}_b^+(\Omega\times(a,b))$ and $\mu_{n,s}, \mu_s \in \mathfrak{M}_s^+(\Omega\times(a,b))$ such that $\{\mu_n\}$ nondecreasingly converges to μ in $\mathfrak{M}_b(\Omega \times (a,b))$. Then, $\{\mu_{n,s}\}$ is nondecreasing and converging to μ_s in $\mathfrak{M}_b(\Omega \times (a,b))$ and there exist decompositions (f,g,h) of μ_0, (f_n,g_n,h_n) of $\mu_{n,0}$ such that $\{f_n\},\{g_n\},\{h_n\}$ strongly converge to f,g,h in $L^1(\Omega \times (a,b)), L^2(\Omega \times (a,b),\mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$ respectively satisfying*

$$||f_n||_{L^1(\Omega\times(a,b))} + \|g_n\|_{L^2(\Omega\times(a,b),\mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega \times (a,b)) \le 2\mu(\Omega \times (a,b)).$$

Remark 4.3.9 *For $0 < \rho \le \frac{1}{3}\min\{\sup_{x\in\Omega} d(x,\partial\Omega), (b-a)^{1/2}\}$, set*

$$\Omega_\rho^j = \{x \in \Omega : d(x,\partial\Omega) > j\rho\} \times (a + (j\rho)^2, a + ((b-a)^{1/2} - j\rho)^2) \quad \text{for} \quad j = 0,...,k_\rho,$$

where $k_\rho = \left[\frac{\min\{\sup_{x\in\Omega} d(x,\partial\Omega),(b-a)^{1/2}\}}{2\rho}\right]$.
We can choose f_n, g_n, h_n in above two Propositions such that for any $j = 1,...,k_\rho$,

$$||f_n||_{L^1(\Omega_\rho^j)} + \|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(\Omega_\rho^j)} \le 2\mu(\Omega_\rho^{j-1}) \ \forall n \in \mathbb{N} \qquad (4.3.9)$$

In fact, set $\mu_j = \chi_{\Omega_\rho^{k_\rho-j}\backslash\Omega_\rho^{k_\rho-j+1}}\mu$ if $j = 1,...,k_\rho - 1$, $\mu_j = \chi_{\Omega\times(a,b)\backslash\Omega_\rho^1}\mu$ if $j = k_\rho$ and $\mu_j = \chi_{\Omega_\rho^{k_\rho}}\mu$ if $j = 0$. From the proof of above two Propositions in [13], for any $\varepsilon > 0$ we can assume supports of f_n, g_n, h_n containing in supp$(\mu) + \tilde{Q}_\varepsilon(0,0)$. Thus, for any $\mu = \mu_j$ we have f_n^j, g_n^j, h_n^j correspondingly such that their supports contain in $\Omega_{\rho,T}^{k_\rho-j-1/2}\backslash\Omega_{\rho,T}^{k_\rho-j+3/2}$ if $j = 1,...,k_\rho - 1$ and $\Omega_T\backslash\Omega_{\rho,T}^{3/2}$ if $j = k_\rho$ and $\Omega_{\rho,T}^{k_\rho-1/2}$ if $j = 0$. By $\mu = \sum_{j=0}^{k_\rho}\mu_j$, thus it is allowed to choose $f_n = \sum_{j=0}^{k_\rho} f_n^j, f_n = \sum_{j=0}^{k_\rho} g_n^j$ and $h_n = \sum_{j=0}^{k_\rho} h_n^j$ and (4.3.9) satisfies since

$$\begin{aligned}
&||f_n||_{L^1(\Omega_\rho^j)} + \|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(\Omega_\rho^j)} \\
&\quad \le \sum_{i=0}^{k_\rho}\Big(||f_n^i||_{L^1(\Omega_\rho^j)} + \|g_n^i\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n^i| + |\nabla h_n^i|||_{L^2(\Omega_\rho^j)}\Big) \\
&\quad = \sum_{i=0}^{k_\rho-j+1}\Big(||f_n^i||_{L^1(\Omega_\rho^j)} + \|g_n^i\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)} + |||h_n^i| + |\nabla h_n^i|||_{L^2(\Omega_\rho^j)}\Big) \\
&\quad \le \sum_{i=j-1}^{k_\rho-j+1} 2\mu_j(\Omega \times (a,b)) = 2\mu(\Omega_\rho^{j-1}).
\end{aligned}$$

Definition 4.3.10 *Let $\mu \in \mathfrak{M}_b(\Omega \times (a,b))$ and $\sigma \in \mathfrak{M}_b(\Omega)$. A measurable function u is a distribution solution to problem (4.3.2) if $u \in L^s(a,b,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and $B(u,\nabla u) \in L^1(\Omega \times (a,b))$ such that*

$$\begin{aligned}
&-\int_{\Omega\times(a,b)} u\varphi_t dxdt + \int_{\Omega\times(a,b)} A(x,t,\nabla u)\nabla\varphi dxdt \\
&\qquad = \int_{\Omega\times(a,b)} B(u,\nabla u)\varphi dxdt + \int_{\Omega\times(a,b)} \varphi d\mu + \int_\Omega \varphi(a)d\sigma
\end{aligned}$$

for every $\varphi \in C_c^1(\Omega \times [a,b))$.

Remark 4.3.11 *Let $\sigma' \in \mathfrak{M}_b(\Omega)$ and $a' \in (a,b)$, set $\omega = \mu + \sigma' \otimes \delta_{\{t=a'\}}$. If u is a distribution solution to problem (4.3.2) with data ω and $\sigma = 0$ such that supp$(\mu) \subset \overline{\Omega} \times [a', b]$, and $u = 0, B(u, \nabla u) = 0$ in $\Omega \times (a, a')$, then $\tilde{u} := u|_{\Omega\times[a',b)}$ is a distribution solution to problem (4.3.2) in $\Omega \times (a', b)$ with data μ and σ'. Indeed, for any $\varphi \in C_c^1(\Omega \times [a', b))$ we defined*

$$\tilde{\varphi}(x,t) = \begin{cases} \varphi(x,t) \text{ if } (x,t) \in \Omega \times [a', b), \\ (1+\varepsilon_0)(t-a')\varphi_t(x,a') + \varphi(x,(1+\varepsilon_0)a' - \varepsilon_0 t) \text{ if } (x,t) \in \Omega \times [a, a'), \end{cases}$$

where $\varepsilon_0 \in \left(0, \frac{b-a'}{a'-a}\right)$.
Clearly, $\tilde{\varphi} \in C_c^1(\Omega \times [a, b))$, thus we have

$$\begin{aligned} -\int_{\Omega\times(a,b)} u\tilde{\varphi}_t dxdt + \int_{\Omega\times(a,b)} A(x,t,\nabla u)\nabla\tilde{\varphi} dxdt \\ = \int_{\Omega\times(a,b)} B(u,\nabla u)\tilde{\varphi} dxdt + \int_{\Omega\times(a,b)} \tilde{\varphi} d\omega, \end{aligned}$$

which implies

$$\begin{aligned} -\int_{\Omega\times(a',b)} \tilde{u}\varphi_t dxdt + \int_{\Omega\times(a',b)} A(x,t,\nabla\tilde{u})\nabla\varphi dxdt \\ = \int_{\Omega\times(a',b)} B(\tilde{u},\nabla\tilde{u})\varphi dxdt + \int_{\Omega\times(a',b)} \varphi d\mu + \int_{\Omega} \varphi(a') d\sigma'. \end{aligned}$$

Definition 4.3.12 *Let $\mu \in \mathfrak{M}(\mathbb{R}^N \times [a, +\infty))$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. A measurable function u is a distribution solution to problem*

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = B(u,\nabla u) + \mu \text{ in } \mathbb{R}^N \times (a, +\infty) \\ u(a) = \sigma \quad \text{ in } \mathbb{R}^N \end{cases} \tag{4.3.10}$$

if $u \in L^s_{loc}(a,\infty, W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and $B(u,\nabla u) \in L^1_{loc}(\mathbb{R}^N \times [a,\infty))$ such that

$$\begin{aligned} -\int_{\mathbb{R}^N\times(a,\infty)} u\varphi_t dxdt + \int_{\mathbb{R}^N\times(a,\infty)} A(x,t,\nabla u)\nabla\varphi dxdt \\ = \int_{\mathbb{R}^N\times(a,\infty)} B(u,\nabla u)\varphi dxdt + \int_{\mathbb{R}^N\times(a,\infty)} \varphi d\mu + \int_{\mathbb{R}^N} \varphi(a) d\sigma \end{aligned}$$

for every $\varphi \in C_c^1(\mathbb{R}^N \times [a,\infty))$.

Definition 4.3.13 *Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. A measurable function u is a distribution solution to problem*

$$u_t - div\,(A(x,t,\nabla u)) = B(u,\nabla u) + \omega \text{ in } \mathbb{R}^{N+1} \tag{4.3.11}$$

if $u \in L^s_{loc}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and $B(u,\nabla u) \in L^1_{loc}(\mathbb{R}^{N+1})$ such that

$$-\int_{\mathbb{R}^{N+1}} u\varphi_t dxdt + \int_{\mathbb{R}^{N+1}} A(x,t,\nabla u)\nabla\varphi dxdt = \int_{\mathbb{R}^{N+1}} B(u,\nabla u)\varphi dxdt + \int_{\mathbb{R}^{N+1}} \varphi d\omega,$$

for every $\varphi \in C_c^1(\mathbb{R}^{N+1})$.

Remark 4.3.14 *Let $\mu \in \mathfrak{M}(\mathbb{R}^N \times [a,+\infty))$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. If u is a distribution solution to problem* (4.3.11) *with data $\omega = \mu + \sigma \otimes \delta_{\{t=a\}}$ such that $u = 0, B(u, \nabla u) = 0$ in $\mathbb{R}^N \times (-\infty, a)$, then $\tilde{u} := u|_{\mathbb{R}^N \times [a,\infty)}$ is a distribution solution to problem* (4.3.10) *in $\mathbb{R}^N \times (a, \infty)$ with data μ and σ, see Remark 4.3.11.*

To prove the existence distribution solution of problem (4.3.10) we need the following results. First, we have local estimates of the renormalized solution which get from [13, Proposition 2.8].

Proposition 4.3.15 *Let u, v be in Definition 4.3.1. There exists $C = C(\Lambda_1, \Lambda_2) > 0$ such that for $k \geq 1$ and $0 \leq \eta \in C_c^\infty(\Omega \times (a, b))$*

$$\int_{|v| \leq k} \eta |\nabla u|^2 dxdt + \int_{|v| \leq k} \eta |\nabla v|^2 dxdt \leq CkA \tag{4.3.12}$$

where

$$A = ||v\eta_t||_{L^1(\Omega\times(a,b))} + |||\nabla u||\nabla \eta|||_{L^1(\Omega\times(a,b))} + ||\eta f||_{L^1(\Omega\times(a,b))} + ||\eta |g|^2||_{L^1(\Omega\times(a,b))}$$
$$+ |||\nabla \eta||g|||_{L^1(\Omega\times(a,b))} + ||\eta|\nabla h|^2||_{L^1(\Omega\times(a,b))} + \int_{\Omega\times(a,b)} \eta d|\mu_s|.$$

For our purpose, we recall the Landes-time approximation of functions w belonging to $L^2(a, b, H_0^1(\Omega))$, introduced in [45], used in [24, 17, 8]. For $\nu > 0$ we define

$$\langle w \rangle_\nu (x, t) = \nu \int_a^{\min\{t,b\}} w(x, s) e^{\nu(s-t)} ds \quad \text{for all } \ (x, t) \in \Omega \times (a, b).$$

We have that $\langle w \rangle_\nu$ converges to w strongly in $L^2(a, b, H_0^1(\Omega))$ and $||\langle w \rangle_\nu||_{L^q(\Omega\times(a,b))} \leq ||w||_{L^q(\Omega\times(a,b))}$ for every $q \in [1, \infty]$. Moreover,

$$(\langle w \rangle_\nu)_t = \nu \left(w - \langle w \rangle_\nu \right) \quad \text{in the sense of distributions}$$

if $w \in L^\infty(\Omega \times (a, b))$ then

$$\int_{\Omega\times(a,b)} (\langle w \rangle_\nu)_t \varphi dxdt = \nu \int_{\Omega\times(a,b)} \left(w - \langle w \rangle_\nu \right) \varphi dxdt \quad \text{for all } \ \varphi \in L^2(a, b, H_0^1(\Omega)).$$

Proposition 4.3.16 *Let $q_0 > 1$ and $0 < \alpha < 1/2$ such that $q_0 > \alpha + 1$. Let $L : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing such that $L(0) = 0$. If u is a solution of*

$$\begin{cases} u_t - div(A(x, t, \nabla u)) + L(u) = \mu \ \text{in} \ \Omega \times (a, b), \\ u = 0 \qquad \text{on} \ \ \partial\Omega \times (a, b), \\ u(a) = 0 \quad \text{in} \ \ \Omega, \end{cases} \tag{4.3.13}$$

with $\mu \in C_c^\infty(\Omega \times (a, b))$ there exists $C_1 > 0$ depending on $\Lambda_1, \Lambda_2, \alpha, q_0$ such that for $0 \leq \eta \in C_c^\infty(D)$ where $D = \Omega' \times (a', b')$, $\Omega' \subset\subset \Omega$ and $a < a' < b' < b$, then

$$\frac{1}{k} \int_D |\nabla T_k(u)|^2 \eta dxdt$$
$$+ \int_D \frac{|\nabla u|^2}{(|u| + 1)^{\alpha+1}} \eta dxdt + |||\nabla u||\nabla \eta|||_{L^1(D)} + ||L(u)\eta||_{L^1(D)} \leq C_1 B, \tag{4.3.14}$$

where $q_1 = \frac{q_0-\alpha-1}{2q_0}$,

$$B = ||\eta_t(|u|+1)||_{L^1(D)} + \int_D (|u|+1)^{q_0}\eta dxdt + \int_D |\nabla\eta^{1/q_1}|^{q_1} dxdt + \int_D \eta d|\mu|.$$

Furthermore, for $T_k(w) \in L^2(a', b', H_0^1(\Omega'))$, *the Landes-time approximation* $\langle T_k(w)\rangle_\nu$ *of the truncate function* $T_k(w)$ *in* D *then for any* $\varepsilon \in (0,1)$ *and* $\nu > 0$

$$\nu \int_D \eta\left(T_k(w) - \langle T_k(w)\rangle_\nu\right) T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu) dxdt$$
$$+ \int_D \eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu) dxdt \le C_2\varepsilon(1+k)B, \qquad (4.3.15)$$

for some $C_2 = C_2(\Lambda_1, \Lambda_2, \alpha, q_0)$.

Proposition 4.3.17 *Let* $q_0 > 1$, $\mu_n = \mu_{n,0} + \mu_{n,s} \in \mathfrak{M}_b(B_n(0) \times (-n^2, n^2))$. *Let* u_n *be a renormalized solution of*

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in } B_n(0) \times (-n^2, n^2), \\ u_n = 0 & \text{on } \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \text{ in } B_n(0), \end{cases} \qquad (4.3.16)$$

relative to the decomposition (f_n, g_n, h_n) *of* $\mu_{n,0}$ *satisfying* (4.3.15) *in Proposition 4.3.16 with* $L \equiv 0$. *Assume that for any* $m \in \mathbb{N}$ *and* $\alpha \in (0, 1/2)$, $D_m := B_m(0) \times (-m^2, m^2)$

$$\frac{1}{k}|||\nabla T_k(u)|^2||_{L^1(D_m)} + |||\nabla u|^2(|u|+1)^{-\alpha-1}||_{L^1(D_m)} + |||\nabla u|||_{L^1(D_m)} + |\mu_n|(D_m)$$
$$+ ||f_n||_{L^1(D_m)} + ||g_n||_{L^2(D_m,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(D_m)} + ||u_n||_{L^{q_0}(D_m)} \le C(m,\alpha)$$

for all $n \ge m$ *and* h_n *is convergent in* $L^1_{loc}(\mathbb{R}^{N+1})$. *Then, there exists a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$ *such that* u_n *converges to* u *a.e in* $\mathbb{R}^{N+1}$ *and in* $L^s_{loc}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ *for any* $s \in [1, \frac{N+2}{N+1})$.

Proofs of above two Propositions are given in the Appendix section. The following result is as a consequence of Proposition 4.3.17.

Corollary 4.3.18 *Let* $\mu_n \in L^1(B_n(0) \times (-n^2, n^2))$. *Let* u_n *be a unique renormalized solution of problem 4.3.16. Assume that for any* $m \in \mathbb{N}$,

$$\sup_{n \ge m} |\mu_n|(B_m(0) \times (-m^2, m^2)) < \infty \quad \text{and} \quad \sup_{n\ge m} \int_{B_m(0)\times(-m^2,m^2)} |u_n|^{q_0} dxdt < \infty.$$

then there exists a subsequence of $\{u_n\}$, *still denoted by* $\{u_n\}$ *such that* u_n *converges to* u *a.e in* $\mathbb{R}^{N+1}$ *and in* $L^s_{loc}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ *for any* $s \in [1, \frac{N+2}{N+1})$.

Finally, we would like to present a technical lemma which will be used several times in the paper, specially in the proof of Theorem 4.2.17, 4.2.19 and 4.2.20. It is a consequence of Vitali Covering Lemma, a proof of lemma can be seen in [22, 21, 54].

Lemma 4.3.19 *Let Ω be a (R_0,δ)- Reifenberg flat domain with $\delta < 1/4$ and let w be an A_∞ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^{L}$ with centers $y_i \in \overline{\Omega}$ and a common radius $r \leq R_0/4$ covers Ω. Set $s_i = T - ir^2/2$ for all $i = 0,1,...,[\frac{2T}{r^2}]$. Let $E \subset F \subset \Omega_T$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that $w(E) < \varepsilon w(\tilde{Q}_r(y_i,s_j))$ for all $i = 1,...,L$, $j = 0,1,...,[\frac{2T}{r^2}]$; and for all $(x,t) \in \Omega_T$, $\rho \in (0,2r]$, we have $\tilde{Q}_\rho(x,t)\cap\Omega_T \subset F$ if $w(E\cap\tilde{Q}_\rho(x,t)) \geq \varepsilon w(\tilde{Q}_\rho(x,t))$. Then $w(E) \leq B\varepsilon w(F)$ for a constant B depending only on N and $[w]_{A_\infty}$.*

Clearly, the Lemma contains the following two Lemmas

Lemma 4.3.20 *Let $0 < \varepsilon < 1, R > 0$ and cylinder $\tilde{Q}_R := \tilde{Q}_R(x_0,t_0)$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ and $w \in A_\infty$. let $E \subset F \subset \tilde{Q}_R$ be two measurable sets in $\mathbb{R}^{N+1}$ with $w(E) < \varepsilon w(\tilde{Q}_R)$ and satisfying the following property : for all $(x,t) \in \tilde{Q}_R$ and $r \in (0,R]$, we have $\tilde{Q}_r(x,t) \cap \tilde{Q}_R \subset F$ provided $w(E \cap \tilde{Q}_r(x,t)) \geq \varepsilon w(\tilde{Q}_r(x,t))$. Then $w(E) \leq B\varepsilon w(F)$ for some $B = B(N,[w]_{A_\infty})$.*

Lemma 4.3.21 *Let $0 < \varepsilon < 1$ and $R > R' > 0$ and let $E \subset F \subset Q = B_R(x_0) \times (a,b)$ be two measurable sets in $\mathbb{R}^{N+1}$ with $|E| < \varepsilon|\tilde{Q}_{R'}|$ and satisfying the following property : for all $(x,t) \in Q$ and $r \in (0,R']$, we have $Q_r(x,t)\cap Q \subset F$ if $|E\cap\tilde{Q}_r(x,t)| \geq \varepsilon|\tilde{Q}_r(x,t)|$. Then $|E| \leq B\varepsilon|F|$ for a constant B depending only on N.*

4.4 Estimates on Potential

In this section, we will develop nonlinear potential theory corresponding to quasilinear parabolic equations.

First we introduce the Wolff parabolic potential of $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{W}_{\alpha,p}^{R}[\mu](x,t) = \int_0^R \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \quad \text{for any} \quad (x,t) \in \mathbb{R}^{N+1},$$

where $\alpha > 0, 1 < p < \alpha^{-1}(N+2)$ and $0 < R \leq \infty$. For convenience, $\mathbb{W}_{\alpha,p}[\mu] := \mathbb{W}_{\alpha,p}^{\infty}[\mu]$.

The following result is an extension of [36, Theorem 1.1], [16, Proposition 2.2] to Parabolic potential.

Theorem 4.4.1 *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$ and $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. There exist constants $C_1, C_2 > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N,\alpha,p,[w]_{A_\infty}$ such that for any $\lambda > 0$ and $\varepsilon \in (0,\varepsilon_0)$*

$$w(\{\mathbb{W}_{\alpha,p}^{R}[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^{R}[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\}) \leq C_1\exp(-C_2\varepsilon^{-1})w(\{\mathbb{W}_{\alpha,p}^{R}[\mu] > \lambda\}) \tag{4.4.1}$$

where $a = 2 + 3^{\frac{N+2-\alpha p}{p-1}}$.

Proof of Theorem 4.4.1. We only consider case $R<\infty$. Let $\{\tilde{Q}_R(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that $\sum_j \chi_{\tilde{Q}_R(x_j,t_j)} \leq M$ in $\mathbb{R}^{N+1}$ for some constant $M=M(N)>0$. It is enough to show that there exist constants $c_1,c_2>0$ and $\varepsilon_0\in(0,1)$ depending on $N,\alpha,p,[w]_{A_\infty}$ such that for any $Q\in\{\tilde{Q}_R(x_j,t_j)\}$, $\lambda>0$ and $\varepsilon\in(0,\varepsilon_0)$

$$w(Q\cap\{\mathbb{W}^R_{\alpha,p}[\mu]>a\lambda,(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}\leq\varepsilon\lambda\})\leq c_1\exp(-c_2\varepsilon^{-1})w(Q\cap\{\mathbb{W}^R_{\alpha,p}[\mu]>\lambda\}). \quad (4.4.2)$$

Fix $\lambda>0$ and $0<\varepsilon<1/10$. We set

$$E=Q\cap\{\mathbb{W}^R_{\alpha,p}[\mu]>a\lambda,(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}\leq\varepsilon\lambda\} \text{ and } F=Q\cap\{\mathbb{W}^R_{\alpha,p}[\mu]>\lambda\}.$$

Thanks to Lemma 4.3.20 we will get (4.4.2) if we verify the following two claims :

$$w(E)\leq c_3\exp(-c_4\varepsilon^{-1})w(Q), \quad (4.4.3)$$

and for any $(x,t)\in Q$, $0<r\leq R$,

$$w(E\cap\tilde{Q}_r(x,t))<c_5\exp(-c_6\varepsilon^{-1})w(\tilde{Q}_r(x,t)), \quad (4.4.4)$$

provided that $\tilde{Q}_r(x,t)\cap Q\cap F^c\neq\emptyset$ and $E\cap\tilde{Q}_r(x,t)\neq\emptyset$, where constants c_3,c_4,c_5 and c_6 depend on N,α,p and $[w]_{A_\infty}$.
Claim (4.4.3) : Set

$$g_k(x,t)=\int_{2^{-k}R}^{2^{-k+1}R}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}.$$

We have for $m\in\mathbb{N}$ and $(x,t)\in E$

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](x,t)&=\sum_{k=m+1}^{\infty}g_k(x,t)+\int_{2^{-m}R}^{R}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}\\&\leq\sum_{k=m+1}^{\infty}g_k(x,t)+m(\mathbb{M}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}}\\&\leq\sum_{k=m+1}^{\infty}g_k(x,t)+m\varepsilon\lambda.\end{aligned}$$

We deduce that for $\beta>0$, $m\in\mathbb{N}$

$$\begin{aligned}|E|&\leq|Q\cap\{\sum_{k=m+1}^{\infty}g_k>(1-m\varepsilon)\lambda\}|\\&=|Q\cap\{\sum_{k=m+1}^{\infty}g_k>\sum_{k=m+1}^{\infty}2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}|\\&\leq\sum_{k=m+1}^{\infty}|Q\cap\{g_k>2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}|.\end{aligned}$$

We can assume that $(x_0,t_0)\in Q$, $(\mathbb{M}^R_{\alpha p}[\mu](x_0,t_0))^{\frac{1}{p-1}}\le\varepsilon\lambda$. Thus, by computing, see [16, Proof of Proposition 2.2] we have for any $k\in\mathbb{N}$

$$|Q\cap\{g_k>s\}|\le\frac{c_7}{s^{p-1}}2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1}.$$

Consequently,

$$\begin{aligned}|E|&\le\sum_{k=m+1}^{\infty}\frac{c_7}{\left(2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\right)^{p-1}}2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1}\\&\le c_7 2^{-(m+1)\alpha p}\left(\frac{\varepsilon}{1-m\varepsilon}\right)^{p-1}|Q|\left(1-2^{-\beta}\right)^{-p+1}\sum_{k=m+1}^{\infty}2^{(\beta(p-1)-\alpha p)(k-m-1)}.\end{aligned}$$

If we choose $\varepsilon^{-1}-2<m\le\varepsilon^{-1}-1$ and $\beta=\beta(\alpha,p)$ so that $\beta(p-1)-\alpha p<0$, we obtain

$$|E|\le c_8\exp(-\alpha p\ln(2)\varepsilon^{-1})|Q|.$$

Thus, we get (4.4.3).
Claim (4.4.4). Take $(x,t)\in Q$ and $0<r\le R$. Now assume that $\tilde{Q}_r(x,t)\cap Q\cap F^c\neq\emptyset$ and $E\cap\tilde{Q}_r(x,t)\neq\emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2)\in\tilde{Q}_r(x,t)\cap Q$ such that $\mathbb{W}^R_{\alpha,p}[\mu](x_1,t_1)\le\lambda$ and $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}}\le\varepsilon\lambda$. We need to prove that

$$w(E\cap\tilde{Q}_r(x,t))<c_9\exp(-c_{10}\varepsilon^{-1})w(\tilde{Q}_r(x,t)).$$

To do this, for all $(y,s)\in E\cap\tilde{Q}_r(x,t)$. $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{3\rho}(x_1,t_1)$ if $\rho>r$.
If $r\le R/3$,

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s)&=\mathbb{W}^r_{\alpha,p}[\mu](y,s)+\int_r^{R/3}\left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}+\int_{R/3}^{R}\left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}\\&\le\mathbb{W}^r_{\alpha,p}[\mu](y,s)+\int_r^{R/3}\left(\frac{\mu(\tilde{Q}_{3\rho}(x_1,t_1))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}+2(\mathbb{M}^R_{\alpha p}[\mu](y,s))^{\frac{1}{p-1}}\\&\le\mathbb{W}^r_{\alpha,p}[\mu](y,s)+3^{\frac{N+2-\alpha p}{p-1}}\lambda+2\varepsilon\lambda.\end{aligned}$$

which follows $\mathbb{W}^r_{\alpha,p}[\mu](y,s)>\lambda$.
If $r\ge R/3$

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s)&\le\mathbb{W}^r_{\alpha,p}[\mu](y,s)+\int_{R/3}^{R}\left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}\\&\le\mathbb{W}^r_{\alpha,p}[\mu](y,s)+2\varepsilon\lambda,\end{aligned}$$

which follows $\mathbb{W}^r_{\alpha,p}[\mu](y,s)>\lambda$.
Thus,

$$w(E\cap\tilde{Q}_r(x,t))\le w(\tilde{Q}_r(x,t)\cap\{\mathbb{W}^r_{\alpha,p}[\mu]>\lambda\}).$$

Since $(x_2,t_2) \in \tilde{Q}_r(x,t)$, $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}} \le \varepsilon\lambda$, so as above we also obtain

$$w(\tilde{Q}_r(x,t) \cap \{\mathbb{W}^r_{\alpha,p}[\mu] > \lambda\}) \le c_9 \exp(-c_{10}\varepsilon^{-1}) w(\tilde{Q}_r(x,t)),$$

which implies (4.4.4). This completes the proof of the Theorem. ∎

Theorem 4.4.2 *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$, $p-1 < q < \infty$ and $0 < s \le \infty$ and $w \in A_\infty$. There holds*

$$C^{-1}||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le ||\mathbb{W}^R_{\alpha,p}[\mu]||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le C||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^{q,s}(\mathbb{R}^{N+1},dw)}, \tag{4.4.5}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ and $R \in (0,\infty]$ where C is a positive constant only depending on N, α, p, q, s and $[w]_{A_\infty}$.

Proof. From (4.4.1) in Theorem (4.4.1), we have for $0 < s < \infty$

$$\begin{aligned}
&||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} = a^s q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&\le c_1 \exp(-c_2\varepsilon^{-1}) q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} + c_3 s \int_0^\infty \lambda^s w(\{(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} > \varepsilon\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&= c_1 \exp(-c_2\varepsilon^{-1}) ||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} + c_3 \varepsilon^{-s} ||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.
\end{aligned}$$

Choose $0 < \varepsilon < \varepsilon_0$ such that $c_1 \exp(-c_2\varepsilon^{-1}) < 1/2$ we get

$$||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} \le c_4 ||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.$$

Similarly, we also get above inequality in case $s = \infty$. So, we proved the right-hand side inequality of (4.4.5).
To complete the proof, we prove the left-hand side inequality of (4.4.5). Since for every $(x,t) \in \mathbb{R}^{N+1}$

$$(\mathbb{W}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}} \le c_5 \left(\mathbb{W}^R_{\alpha,p}[\mu](x,t) + \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \right) \quad \text{and}$$

$$\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \le c_6 \mathbb{W}^R_{\alpha,p}[\mu](x,t),$$

thus it is enough to show that for any $\lambda > 0$

$$w\left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > \lambda \right\} \right) \le c_7 w\left(\left\{ (x,t) : \left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} > c_8\lambda \right\} \right). \tag{4.4.6}$$

Let $\{Q_j\} = \{\tilde{Q}_{R/4}(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that for any $Q_j \in \{Q_j\}$, there exist $Q_{j,1}, ..., Q_{j,M_1} \in \{Q_j\}$ with $\sum_j \sum_{k=1}^{M_1} \chi_{Q_{j,k}} \le M_2$ and $Q_j + \tilde{Q}_{2R}(0,0) \subset \bigcup_{k=1}^{M_1} Q_{j,k}$ for some integer constants $M_i = M_i(N), i = 1, 2$. Then,

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right)\leq\sum_j w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\cap Q_j\right)$$
$$\leq\sum_j w\left(\left\{(x,t):\sum_{k=1}^{M_1}\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}>\lambda^{p-1}\right\}\cap Q_j\right)$$
$$\leq\sum_j\sum_{k=1}^{M_1} w\left(\left\{(x,t):\left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}\cap Q_j\right)$$
$$=\sum_j\sum_{k=1}^{M_1} a_{j,k}w(Q_j),$$

where $a_{j,k}=1$ if $\left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda$ and $a_{j,k}=0$ if otherwise.
Using the strong doubling property of w, there is $c_9=c_9(N,[w]_{A_\infty})$ such that $w(Q_j)\leq c_9w(Q_{j,k})$. On the other hand, if $a_{j,k}=1$ then $Q_{j,k}\subset\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}$.
Therefore,

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right)\leq\sum_j\sum_{k=1}^{M_1}c_9a_{j,k}w(Q_{j,k})$$
$$\leq\sum_j\sum_{k=1}^{M_1}c_9w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}\cap Q_{j,k}\right),$$

which implies (4.4.6) since $\sum_j\sum_{k=1}^{M_1}\chi_{Q_{j,k}}\leq M_2$ in $\mathbb{R}^{N+1}$. ■

Theorem 4.4.3 *Let* $0<\alpha p<N+2$ *and* $w\in A_\infty$ *There exist* $C_1,C_2>0$ *depending on* N,α,p *and* $[w]_{A_\infty}$ *such that for any* $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$, *any cylinder* $\tilde{Q}_\rho\subset\mathbb{R}^{N+1}$ *there holds*

$$\frac{1}{w(\tilde{Q}_{2\rho})}\int_{\tilde{Q}_{2\rho}}\exp\left(C_1\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t)\right)dw(x,t)\leq C_2 \tag{4.4.7}$$

provided $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)}\leq 1$, *where* $\mu_{\tilde{Q}_\rho}=\chi_{\tilde{Q}_\rho}\mu$.

Proof. Assume that $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)}\leq 1$. We apply Theorem (4.4.1) to $\mu_{\tilde{Q}_\rho}$. Then, choose $\varepsilon=\lambda^{-1}$ for all $\lambda\geq\lambda_0:=\max\{\varepsilon_0^{-1},\frac{N+2-\alpha p}{p-1}\}$, we obtain

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>a\lambda\}\cap\tilde{Q}_{2\rho})\leq C_1\exp(-C_2\varepsilon^{-1})w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]>\lambda\})\ \ \forall\ \lambda\geq\lambda_0,$$

On the other hand, if $\rho>R$, clearly we have $\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}]\equiv 0$ in $\mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$, if $\rho\leq R$, for any $(x,t)\in\mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$

$$\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t)=\int_\rho^R\left(\frac{\mu_{\tilde{Q}_\rho}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}\leq\frac{N+2-\alpha p}{p-1}\left(\frac{\mu(\tilde{Q}_\rho)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\leq\lambda_0.$$

So, we get $\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > \lambda\} \subset \tilde{Q}_{2\rho}$ for all $\lambda \geq \lambda_0$. This can be written under the form

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > a\lambda\} \cap \tilde{Q}_{2\rho}) \leq \left(\chi_{(0,t_0]} + C_1 \exp(-C_2\lambda)\right) w(\tilde{Q}_{2\rho}),$$

for all $\lambda > 0$. Therefore, we get (4.4.7). ■

In what follows, we need some estimates on Wolff parabolic potential :

Proposition 4.4.4 *Let $p > 1, 0 < \alpha p < N+2$ and $q > 1, \alpha pq < N+2$. There exist C_1, C_2 such that*

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{(N+2)(p-1)}{N+2-\alpha p},\infty}(\mathbb{R}^{N+1})} \leq C_1(\mu(\mathbb{R}^{N+1}))^{\frac{1}{p-1}} \quad \forall\ \mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1}), \tag{4.4.8}$$

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq},\infty}(\mathbb{R}^{N+1})} \leq C_2||\mu||^{\frac{1}{p-1}}_{L^{q,\infty}(\mathbb{R}^{N+1})} \quad \forall\ \mu \in L^{q,\infty}(\mathbb{R}^{N+1}), \mu \geq 0, \tag{4.4.9}$$

and

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq}}(\mathbb{R}^{N+1})} \leq C_2||\mu||^{\frac{1}{p-1}}_{L^{q}(\mathbb{R}^{N+1})} \quad \forall\ \mu \in L^{q}(\mathbb{R}^{N+1}), \mu \geq 0. \tag{4.4.10}$$

In particular, for $s > \frac{(p-1)(N+2)}{N+2-\alpha p}$, we define $F(\mu) := (\mathbb{W}_{\alpha,p}[\mu])^s$ for all $\mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1})$. Then,

$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})} \leq C_3||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})} \quad \text{and}$$

$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})} \leq C_3||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})},$$

for some constant $C_i = C_i(N,p,\alpha,s)$ for $i = 3,4$.

Proof. Let $s \geq 1$ such that $\alpha sp < N+2$. It is known that if $\mu \in L^{s,\infty}(\mathbb{R}^{N+1})$ then

$$|\mu|(\tilde{Q}_\rho(x,t)) \leq c_1||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}\rho^{\frac{N+2}{s'}} \quad \forall \ \ \rho > 0.$$

Thus for $\delta = ||\mu||^{\frac{s}{N+2}}_{L^{s,\infty}(\mathbb{R}^{N+1})} (\mathbb{M}(\mu)(x,t))^{-\frac{s}{N+2}}$ we have

$$\begin{aligned}
\mathbb{W}_{\alpha,p}[\mu](x,t) &= \int_0^\delta \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + \int_\delta^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\
&\leq c_2\left(\mathbb{M}(\mu)(x,t)\right)^{\frac{1}{p-1}} \delta^{\frac{\alpha p}{p-1}} + c_2||\mu||^{\frac{1}{p-1}}_{L^{s,\infty}(\mathbb{R}^{N+1})}\delta^{-\frac{N+2-\alpha sp}{s(p-1)}} \\
&= c_3\left(\mathbb{M}(\mu)(x,t)\right)^{\frac{N+2-\alpha sp}{(p-1)(N+2)}} ||\mu||^{\frac{\alpha sp}{(p-1)(N+2)}}_{L^{s,\infty}(\mathbb{R}^{N+1})}.
\end{aligned}$$

So, for any $\lambda > 0$

$$|\{\mathbb{W}_{\alpha,p}[\mu] > \lambda\}| \leq |\{\mathbb{M}(\mu) > c_4||\mu||^{-\frac{\alpha sp}{N+2-\alpha sp}}_{L^{s,\infty}(\mathbb{R}^{N+1})}\lambda^{\frac{(p-1)(N+2)}{N+2-\alpha sp}}\}|.$$

Hence, since $\mathbb{M}$ is bounded from $\mathfrak{M}^+_b(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^q(\mathbb{R}^{N+1})$ ($L^{q,\infty}(\mathbb{R}^{N+1})$ resp.) to itself, we get the result. ■

Remark 4.4.5 *Assume that $\alpha p = N+2$ and $R>0$. As above we also have for any $\varepsilon>0$*

$$\mathbb{W}^R_{\alpha,p}[\mu](x,t) \leq C_{1,\varepsilon} \max\left\{ (|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}, \left((\mathbb{M}(\mu)(x,t))^{\varepsilon} (|\mu|(\mathbb{R}^{N+1}))^{\frac{\alpha p}{p-1}} R^{\varepsilon\alpha p}\right)^{\frac{1}{\alpha p+\varepsilon(p-1)}} \right\}$$

where $C_{1,\varepsilon} = C_1(N,\alpha,p,\varepsilon)$.
Therefore, for any $\lambda > C_\varepsilon (|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}$,

$$|\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\}| \leq C_{2,\varepsilon} \left(\frac{(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}}{\lambda} \right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} R^{\alpha p}, \tag{4.4.11}$$

where $C_{2,\varepsilon} = C_2(N,\alpha,p,\varepsilon)$. In particular, if $\mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1})$ then $\mathbb{W}^R_{\alpha,p}[\mu] \in L^s_{loc}(\mathbb{R}^{N+1})$ for all $s>0$.

Remark 4.4.6 *Assume that $p,q>1, 0<\alpha pq<N+2$. As in [59, Theorem 3], it is easy to prove that if $w \in A_{\frac{q(N+2-\alpha)}{N+2-\alpha pq}}$, i.e, $0<w\in L^1_{loc}(\mathbb{R}^{N+1})$ and for any $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$*

$$\sup_{\tilde{Q}_\rho(y,s)\subset \mathbb{R}^{N+1}} \left(\left(\fint_{\tilde{Q}_\rho(y,s)} w dxdt \right) \left(\fint_{\tilde{Q}_\rho(y,s)} w^{-\frac{N+2-\alpha pq}{(q-1)(N+2)}} dxdt \right)^{\frac{(q-1)(N+2)}{N+2-\alpha pq}} \right) = C_1 < \infty,$$

then

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{M}_{\alpha p}[|f|])^{\frac{(N+2)q}{N+2-\alpha pq}} w dxdt \right)^{\frac{N+2-\alpha pq}{(N+2)q}} \leq C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^q w^{1-\frac{\alpha pq}{N+2}} dxdt \right)^{\frac{1}{q}},$$

for some a constant $C_2 = C_2(N,\alpha p,q,C_1)$.
Therefore, from (4.4.5) in Theorem 4.4.2 we get a weighted version of (4.4.10)

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{W}_{\alpha,p}[|f|])^{\frac{(N+2)(p-1)q}{N+2-\alpha pq}} w dxdt \right)^{\frac{N+2-\alpha pq}{(N+2)q}} \leq C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^p w^{1-\frac{\alpha p}{N+2}} dxdt \right)^{\frac{1}{p}}.$$

The following another version of (4.4.10) in the Lorentz-Morrey spaces involving calorie.

Proposition 4.4.7 *Let $p,q>1$, and $0<\alpha pq<\theta \leq N+2$. There exists a constant $C>0$ such that*

$$||\, (\mathbb{W}_{\alpha,p}[|\mu|])^{p-1}\, ||_{L^{\frac{\theta q}{\theta-\alpha pq};\theta}(\mathbb{R}^{N+1})} \leq C ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} \quad \forall \mu \in L^{q;\theta}(\mathbb{R}^{N+1}). \tag{4.4.12}$$

Proof. As the proof of Proposition 4.4.4 we have

$$\mathbb{W}_{\alpha,p}[|\mu|] \leq c_1 \left(\mathbb{M}_{\theta/q}[|\mu|]\right)^{\frac{\alpha pq}{\theta(p-1)}} (\mathbb{M}[|\mu|])^{\frac{\theta-\alpha pq}{\theta(p-1)}}.$$

Since $\mathbb{M}_{\theta/q}[|\mu|] \leq c_2 (\mathbb{M}_\theta[|\mu|^q])^{1/q}$, above inequality becomes

$$\mathbb{W}_{\alpha,p}[\mu] \leq c_3 (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha p}{\theta(p-1)}} (\mathbb{M}[\mu])^{\frac{\theta-\alpha pq}{\theta(p-1)}}. \tag{4.4.13}$$

Take $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$, we have

$$\begin{aligned}\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt &\le c_4 \Bigg(\int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu] \right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\quad + \int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{(\tilde{Q}_{2\rho}(y,s))^c}\mu] \right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \Bigg) \\ &= A + B.\end{aligned}$$

Using inequality (4.4.13) and boundless $\mathbb{M}$ from $L^q(\mathbb{R}^{N+1})$ to itself, yield

$$\begin{aligned}A &\le c_5 \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha q}{\theta-\alpha pq}} \left(\mathbb{M}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu] \right)^q dxdt \\ &\le c_6 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\alpha q^2}{\theta-\alpha pq}} \int_{\chi_{\tilde{Q}_{2\rho}(y,s)}} |\mu|^q dxdt \\ &\le c_7 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

On the other hand, since $|\mu|(\tilde{Q}_r(x,t)) \le c_8 ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{N+2-\frac{\theta}{q}}$ for all $\tilde{Q}_r(x,t) \subset \mathbb{R}^{N+1}$,

$$\begin{aligned}B &\le \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(\frac{|\mu|(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\le c_9 \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{-\frac{\theta}{q}+\alpha} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\le c_{10} ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

Therefore,

$$\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \le c_{11} ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta},$$

which follows (4.4.12). ■

In the next result we state a series of equivalent norms concerning potentials $\mathbb{I}_\alpha[\mu], \mathbb{I}_\alpha^R[\mu], \mathcal{H}_\alpha[\mu], \mathcal{G}_\alpha[\mu]$.

Proposition 4.4.8 *Let $q > 1$, $0 < \alpha < N+2$ and $R > 0$. There exist constants $C_1 = C_1(N,\alpha,q)$ and $C_2 = C_2(N,\alpha,q,R)$ such that the following statements hold*

a. *for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$*

$$C_1^{-1} ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\mathcal{H}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_1 ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \quad \text{and} \tag{4.4.14}$$

$$C_1^{-1} ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\check{\mathcal{H}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_1 ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{4.4.15}$$

b. *for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$*

$$C_2^{-1} ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_2 ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \quad \text{and} \tag{4.4.16}$$

$$C_2^{-1} ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} \le ||\check{\mathcal{G}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \le C_2 ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{4.4.17}$$

where $\overset{\vee}{\mathcal{H}}_\alpha[\mu]$ *is the backward parabolic Riesz potential, defined by*

$$\overset{\vee}{\mathcal{H}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{H}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{H}_\alpha(x-y,s-t)d\mu(y,s),$$

and $\overset{\vee}{\mathcal{G}}_\alpha[\mu]$ *is the backward parabolic Bessel potential :*

$$\overset{\vee}{\mathcal{G}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{G}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{G}_\alpha(y-x,s-t)d\mu(y,s).$$

Proof. a. We have :

$$\frac{c_1^{-1}}{t^{\frac{N+2-\alpha}{2}}}\chi_{t>0}\chi_{|x|\le 2\sqrt{t}} \le \mathcal{H}_\alpha(x,t) \le \frac{c_1}{\max\{|x|,\sqrt{2|t|}\}^{N+2-\alpha}},$$

which implies

$$c_2^{-1}\int_0^\infty \frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r} \le \mathcal{H}_\alpha(x,t) \le c_2\int_0^\infty \frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}.$$

Thus,

$$c_2^{-1}\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r} \le \mathcal{H}_\alpha[\mu](x,t) \le c_2\mathbb{I}_\alpha[\mu](x,t). \tag{4.4.18}$$

Thanks to Theorem 4.4.2 we will finish the proof of (4.4.14) when we show that

$$\int_{\mathbb{R}}\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \ge c_3\int_{\mathbb{R}}\int_0^{+\infty}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}dt.$$

Indeed, we have for $r_k = (\frac{2}{\sqrt{3}})^{-k}$,

$$\begin{aligned}
&\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2,t-r^2/4)\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q \\
&\qquad \ge c_4\left(\sum_{k=-\infty}^{\infty}\frac{\mu\left(B(x,r_k)\times(t-r_k^2,t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q \\
&\qquad \ge c_4\sum_{k=-\infty}^{\infty}\left(\frac{\mu\left(B(x,r_k)\times(t-r_k^2,t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_0^{\infty}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{1}{4}r^2)\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\\
&\quad\geq c_4\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\mu\left(B(x,r_k)\times(t-r_k^2,t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q dt\\
&\quad= c_4\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\mu\left(B(x,r_k)\times(t-\frac{1}{3}r_k^2,t+\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}}\right)^q dt\\
&\quad\geq c_5\int_{\mathbb{R}}\int_0^{+\infty}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}dt.
\end{aligned}$$

Similarly, we also can prove (4.4.15).

b. Obviously

$$\begin{aligned}
&\frac{c_6^{-1}\exp(-4R^2)}{t^{\frac{N+2-\alpha}{2}}}\chi_{0<t<4R^2}\chi_{|x|\leq 2\sqrt{t}}\leq\mathcal{G}_\alpha(x,t)\\
&\leq\frac{c_6}{\max\{|x|,\sqrt{2|t|}\}^{N+2-\alpha}}\chi_{\tilde{Q}_{R/2}(0,0)}(x,t)+\frac{c_6}{R^{N+2-\alpha}}\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right).
\end{aligned}$$

Thus, we can assert that

$$\begin{aligned}
c_7(R)\int_0^{2R}\frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}\leq\mathcal{G}_\alpha(x,t)\leq c_8\int_0^R\frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}\\
+c_9(R)\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\chi_{\tilde{Q}_{R/2}(0,0)}(x-y,t-s)dyds.
\end{aligned}$$

Immediately, we get

$$c_7(R)\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\leq\mathcal{G}_\alpha[\mu](x,t)\leq c_8\mathbb{I}_\alpha^R[\mu](x,t)+c_9(R)F(x,t),\tag{4.4.19}$$

where $F(x,t)=\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right)dyds$.

As above, we can show that

$$\int_0^{\infty}\left(\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\geq c_{10}\int_0^{\infty}\int_0^R\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}.$$

Thus, thanks to Theorem 4.4.2 we get the left-hand side inequality of (4.4.16).

To show the right-hand side of (4.4.16), we use $\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right)\leq c_{10}R^{-(N+2-\alpha)}\mathbb{I}_\alpha^R[\mu](x-y,t-s)$ and Young inequality

$$\begin{aligned}
&||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}\leq c_8||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}+c_9(R)||F||_{L^q(\mathbb{R}^{N+1})}\\
&\quad\leq c_8||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}+c_{11}(R)||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right)dxdt\\
&\quad=c_{12}(R)||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}.
\end{aligned}$$

Similarly, we also can prove (4.4.17). This completes the proof of the Proposition. ■

Remark 4.4.9 *Assume that* $0 < \alpha < N+2$. *From* (4.4.8) *in Proposition 4.4.4 and* $||\mathcal{G}_\alpha[\mu]||_{L^1(\mathbb{R}^{N+1})} \leq c_1\mu(\mathbb{R}^{N+1})$ *we deduce that for* $1 \leq s < \frac{N+2}{N+2-\alpha}$

$$||\mathcal{G}_\alpha[\mu]||_{L^s(\mathbb{R}^{N+1})} \leq c_2\mu(\mathbb{R}^{N+1}) \ \ \forall\, \mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$$

Next, we introduce the following kernel :

$$E_\alpha^R(x,t) = \max\{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)}\chi_{\tilde{Q}_R(0,0)}(x,t)$$

where $0 < \alpha < N+2$ and $0 < R \leq \infty$. We denote E_α^∞ by E_α. It is easy to see that $E_\alpha * \mu = (N+2-\alpha)\mathbb{I}_\alpha[\mu]$ and $||E_\alpha^R * \mu||_{L^s(\mathbb{R}^{N+1})}$ is equivalent to $||\mathbb{I}_\alpha^R[\mu]||_{L^s(\mathbb{R}^{N+1})}$ for every $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ where $1 \leq s < \infty$.
We obtain equivalences of capacities $\text{Cap}_{E_\alpha,p}, \text{Cap}_{E_\alpha^R,p}, \text{Cap}_{\mathcal{H}_\alpha,p}$ and $\text{Cap}_{\mathcal{G}_\alpha,p}$.

Corollary 4.4.10 *Let* $p > 1$, $1 < \alpha < N+2$ *and* $R > 0$. *There exist constants* $C_1 = C_1(N,\alpha,p)$ *and* $C_2 = C_2(N,\alpha,p,R)$ *such that the following statements hold*
a. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$C_1^{-1}\,Cap_{\mathcal{H}_\alpha,p}(E) \leq Cap_{E_\alpha,p}(E) \leq C_1\,Cap_{\mathcal{H}_\alpha,p}(E) \tag{4.4.20}$$

b. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$C_2^{-1}\,Cap_{\mathcal{G}_\alpha,p}(E) \leq Cap_{E_\alpha^R,p}(E) \leq C_2\,Cap_{\mathcal{G}_\alpha,p}(E) \tag{4.4.21}$$

c. *for any compact* $E \subset \mathbb{R}^{N+1}$

$$Cap_{\mathcal{H}_\alpha,p}(E) \leq Cap_{\mathcal{G}_\alpha,p}(E) \leq C_1\left(Cap_{\mathcal{H}_\alpha,p}(E) + \left(Cap_{\mathcal{H}_\alpha,p}(E)\right)^{\frac{N+2}{N+2-\alpha p}}\right) \tag{4.4.22}$$

provided $1 < \alpha p < N+2$.

Proof. By [2, Chapter 2], we have

$$\begin{aligned}
&\text{Cap}_{E_\alpha,p}(E)^{1/p} = \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||E_\alpha * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\},\\
&\text{Cap}_{E_\alpha^R,p}(E)^{1/p} = \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||E_\alpha^R * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\},\\
&\text{Cap}_{\mathcal{H}_\alpha,p}(E)^{1/p} = \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\} \quad \text{and}\\
&\text{Cap}_{\mathcal{G}_\alpha,p}(E)^{1/p} = \sup\{\mu(E) : \mu \in \mathfrak{M}^+(E), ||\overset{\vee}{\mathcal{G}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})} \leq 1\}.
\end{aligned}$$

Thanks to (4.4.15), (4.4.17) in Proposition 4.4.8 and $\mathbb{I}_\alpha[\mu] = E_\alpha * \mu$ and $||E_\alpha^R * \mu||_{L^s(\mathbb{R}^{N+1})}$ is equivalent to $||\mathbb{I}_\alpha^R[\mu]||_{L^s(\mathbb{R}^{N+1})}$, we get (4.4.20) and (4.4.21).
Since $\mathcal{G}_\alpha \leq \mathcal{H}_\alpha$, thus $\text{Cap}_{\mathcal{H}_\alpha,p}(E) \leq \text{Cap}_{\mathcal{G}_\alpha,p}(E)$ for any compact $E \subset \mathbb{R}^{N+1}$. Put $\text{Cap}_{E_\alpha,p}(E) = a > 0$. We need to prove that

$$\text{Cap}_{E_\alpha^1,p}(E) \leq c_1\left(a + a^{\frac{N+2}{N+2-\alpha p}}\right). \tag{4.4.23}$$

We will follow a proof of Yu.V. Netrusov in [2, Chapter 5]. First, we can find $f \in L^p_+(\mathbb{R}^{N+1})$ such that $||f||_{L^p(\mathbb{R}^{N+1})} \leq 2a$ and $E_\alpha * f \geq \chi_E$. Set $F_\alpha = E_\alpha - E^1_\alpha$, we have $c_2 F_\alpha \leq E^1_\alpha * F_\alpha$ for some $c_1 > 0$. Thus, $E \subset \{E^1_\alpha * f \geq 1/2\} \cup \{E^1_\alpha * (F_\alpha * f) \geq c_2/2\}$.
Since $||E^1_\alpha||_{L^1(\mathbb{R}^{N+1})} < \infty$, for $c_3 = c_2(4||E^1_\alpha||_{L^1(\mathbb{R}^{N+1})})^{-1}$

$$E^1_\alpha * (F_\alpha * f) \leq c_2/4 + E^1_\alpha * g \text{ with } g = \chi_{F_\alpha * f \geq c_3} F_\alpha * f,$$

which follows $E \subset \{E^1_\alpha * f \geq 1/2\} \cup \{E^1_\alpha * g \geq c_2/4\}$.
Using the subadditivity of capacity, we have

$$\begin{aligned}\mathrm{Cap}_{E^1_\alpha,p}(E) &\leq \mathrm{Cap}_{E^1_\alpha,p}(\{E^1_\alpha * f \geq 1/2\}) + \mathrm{Cap}_{E^1_\alpha,p}(\{E^1_\alpha * g \geq c_1/4\})\\ &\leq 2^p ||f||^p_{L^p(\mathbb{R}^{N+1})} + (4/c_1)^p ||g||^p_{L^p(\mathbb{R}^{N+1})}\\ &\leq 2^p ||f||^p_{L^p(\mathbb{R}^{N+1})} + (4/c_1)^p c_3^{p*-p} ||E_\alpha * f||^{p*}_{L^{p*}(\mathbb{R}^{N+1})}, \text{ with } p* = \frac{(N+2)p}{N+2-\alpha p}.\end{aligned}$$

On the other hand, from (4.4.10) in Proposition 4.4.4 we have

$$||E_\alpha * f||_{L^{p*}(\mathbb{R}^{N+1})} \leq c_4 ||f||_{L^p(\mathbb{R}^{N+1})}.$$

Hence, we get (4.4.23). ∎

Remark 4.4.11 *Since $\mathcal{G}_\alpha \in L^1(\mathbb{R}^{N+1})$,*

$$\int_{\mathbb{R}^{N+1}} (\mathcal{G}_\alpha * f)^p \, dxdt \leq ||\mathcal{G}_\alpha||^p_{L^1(\mathbb{R}^{N+1})} \int_{\mathbb{R}^{N+1}} f^p dxdt \quad \forall f \in L^p_+(\mathbb{R}^{N+1})$$

Thus, for any Borel set $E \subset \mathbb{R}^{N+1}$

$$Cap_{\mathcal{G}_\alpha,p}(E) \geq C|E| \text{ with } C = ||\mathcal{G}_\alpha||^{-p}_{L^1(\mathbb{R}^{N+1})}. \tag{4.4.24}$$

Remark 4.4.12 *It is well-known that $\mathcal{H}_2$ is the fundamental solution of the heat operator $\frac{\partial}{\partial t} - \Delta$. In [31], R. Gariepy and W. P. Ziemer introduced the following capacity :*

$$C_{\mathcal{H}_2}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}^+(K), \mathcal{H}_2[\mu] \leq 1\},$$

whenever $K \subset \mathbb{R}^{N+1}$ is compact. Thanks to [2, Theorem 2.5.5], we obtain

$$Cap_{\mathcal{H}_1,2}(K) = C_{\mathcal{H}_2}(K).$$

Remark 4.4.13 *For any Borel set $E \subset \mathbb{R}^N$, then we always have $Cap_{\mathcal{G}_1,2}(E \times \{t=0\}) = 0$ In fact,*

$$Cap_{E^1_1,2}(B_1(0) \times \{t=0\}) = \sup\{\omega(B_1(0)) : \omega \in \mathfrak{M}^+(B_1(0)), ||E^1_1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} \leq 1\}.$$

Since $||E^1_1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} = \infty$ if $\omega \neq 0$, thus $Cap_{\mathcal{G}_1,2}(B_1(0) \times \{t=0\}) = Cap_{E^1_1,2}(B_1(0) \times \{t=0\}) = 0$. In particular, $Cap_{\mathcal{G}_1,2}$ is not absolutely continuous with respect to capacity $C_{1,2}(.,\Omega \times (a,b))$. This capacity will be defined in next section.

Remark 4.4.14 *Let $p > 1$ and $\alpha > 0$. Case $\alpha p \geq p+1$, we always have $||\mathcal{H}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^N)} = \infty$ for any $\mu \in \mathfrak{M}^+(\mathbb{R}^N)\backslash\{0\}$ which implies $Cap_{\mathcal{H}_\alpha,p}(\tilde{Q}_1(0,0)) = 0$. If $0 < \alpha p < N+2$, $Cap_{\mathcal{H}_\alpha,p}(\tilde{Q}_\rho(0,0)) = c\rho^{N+2-\alpha p}$ for some constant c. From (4.4.22) in Corollary 4.4.10 we get $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \approx \rho^{N+2-\alpha p}$ for $0 < \rho < 1$ if $\alpha p < N+2$. Since $||\mathcal{G}_\alpha[\delta_{(0,0)}]||_{L^{p'}(\mathbb{R}^{N+1})} < \infty$ thus $Cap_{\mathcal{G}_\alpha,p}((0,0)) > 0$ if $\alpha p > N+2$.*
If $\alpha p = N+2$, $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \approx (\log(1/\rho))^{1-p}$ for any $0 < \rho < 1/2$. In fact, we can prove that $||\mathbb{I}_\alpha^{1/2}[\mu]||_{L^{p'}(\mathbb{R}^N)} \leq c_1$ for any $d\mu(x,t) = (log(1/\rho))^{-1/p'}\rho^{-N-2}\chi_{\tilde{Q}_\rho(0,0)}dxdt$ it follows $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \geq c_2(\log(1/\rho))^{1-p}$. Moreover, for $\mu \in \mathfrak{M}^+(\tilde{Q}_\rho)$, if $||\mathbb{I}_\alpha^3[\mu]||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \leq 1$,

$$
\begin{aligned}
1 &\geq \int_{\tilde{Q}_1(0,0)\backslash\tilde{Q}_\rho(0,0)} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^{p'} dxdt \\
&\geq \int_{\tilde{Q}_1(0,0)\backslash\tilde{Q}_\rho(0,0)} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{1}{r^{N+2-\alpha}}\frac{dr}{r}\right)^{p'} dxdt\mu(\tilde{Q}_\rho(0,0))^{p'} \\
&\geq c_3\log(1/\rho)\mu(\tilde{Q}_\rho(0,0))^{p'}.
\end{aligned}
$$

So $Cap_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho(0,0)) \leq c_4\mu(\tilde{Q}_\rho(0,0))^p \leq c_5(\log(1/\rho))^{1-p}$.

Definition 4.4.15 *The parabolic Bessel potential $\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})$, $\alpha > 0$ and $p > 1$ is defined by*

$$\mathcal{L}^p_\alpha(\mathbb{R}^{N+1}) = \{f : f = \mathcal{G}_\alpha * g, g \in L^p(\mathbb{R}^{N+1})\} \tag{4.4.25}$$

with the norm $||f||_{\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})} := ||g||_{L^p(\mathbb{R}^{N+1})}$. We denote its dual space by $\left(\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})\right)^*$.

Definition 4.4.16 *For k a positive integer, the Sobolev space $W^{2k,k}_p(\mathbb{R}^{N+1})$ is defined by*

$$W^{2k,k}_p(\mathbb{R}^{N+1}) = \{\varphi : \frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i} \in L^p(\mathbb{R}^{N+1}) \text{ for any } i_1+...+i_N+2i \leq 2k\}$$

with the norm

$$||\varphi||_{W^{2k,k}_p(\mathbb{R}^{N+1})} = \sum_{i_1+...+i_N+2i\leq 2k} ||\frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i}||_{L^p(\mathbb{R}^{N+1})}.$$

We denote its dual space by $\left(W^{2k,k}_p(\mathbb{R}^{N+1})\right)^*$. We also define a corresponding capacity on compact set $E \subset \mathbb{R}^{N+1}$,

$$Cap_{2k,k,p}(E) = \inf\{||\varphi||^p_{W^{2k,k}_p(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } E\}.$$

Let us recall Richard J. Bagby's result, proved in [4].

Theorem 4.4.17 *Let $p > 1$ and k be a positive integer. Then, there exists a constant C depending on N, k, p such that for any $u \in \mathcal{L}^p_{2k}(\mathbb{R}^{N+1})$,*

$$C^{-1}||u||_{W^{2k,k}_p(\mathbb{R}^{N+1})} \leq ||u||_{\mathcal{L}^p_{2k}(\mathbb{R}^{N+1})} \leq C||u||_{W^{2k,k}_p(\mathbb{R}^{N+1})}.$$

Above Theorem gives the assertion of equivalence of capacity $\mathrm{Cap}_{2k,k,p}, \mathrm{Cap}_{\mathcal{G}_{2k},p}$.

Corollary 4.4.18 *Let $p>1$ and k be a positive integer. There exists a constant C depending on N,k,p such that for any compact set $E\subset\mathbb{R}^{N+1}$*

$$C^{-1}\,Cap_{2k,k,p}(E)\leq Cap_{\mathcal{G}_{2k},p}(E)\leq C\,Cap_{2k,k,p}(E). \tag{4.4.26}$$

Next result provides some relations of Riesz, Bessel parabolic potential and Riesz, Bessel potential.

Proposition 4.4.19 *Let $q>1$ and $\frac{2}{q'}<\alpha<N+\frac{2}{q'}$. There exists a constant C depending on N,q,α such that for any $\omega\in\mathfrak{M}^+(\mathbb{R}^N)$*

$$\begin{aligned}&C^{-1}||\mathbf{I}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\\&\quad\leq||\mathcal{H}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})},||\overset{\vee}{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})}\leq C||\mathbf{I}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\end{aligned} \tag{4.4.27}$$

and

$$\begin{aligned}&C^{-1}||\mathbf{G}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\\&\quad\leq||\mathcal{G}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})},||\overset{\vee}{\mathcal{G}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})}\leq C||\mathbf{G}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}\end{aligned} \tag{4.4.28}$$

where $\delta_{\{t=0\}}$ is the Dirac mass in time at 0.

Proof. We have

$$\mathbb{I}_\alpha[\omega\otimes\delta_{\{t=0\}}](x,t)=\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r},\ \mathbb{I}_\alpha^1[\omega\otimes\delta_{\{t=0\}}](x,t)=\int_{\min\{1,\sqrt{2|t|}\}}^1\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}.$$

By [16, Theorem 2.3] and Proposition 4.4.8, thus it is enough to show that

$$c_1^{-1}\int_0^\infty\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\leq\int_{\mathbb{R}}\left(\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\leq c_1\int_0^\infty\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}, \tag{4.4.29}$$

and

$$\begin{aligned}&c_1^{-1}\int_0^{1/2}\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\\&\quad\leq\int_{\mathbb{R}}\left(\int_{\min\{1,\sqrt{2|t|}\}}^1\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt\leq c_1\int_0^1\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-2/q}}\right)^q\frac{dr}{r}\end{aligned} \tag{4.4.30}$$

Indeed, by changing of variables

$$\int_{-\infty}^\infty\left(\int_{\sqrt{2|t|}}^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt=2\int_0^\infty t\left(\int_t^\infty\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt. \tag{4.4.31}$$

Using Hardy's inequality, we have

$$\int_0^\infty t\left(\int_t^\infty \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \leq c_2\int_0^\infty r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr$$

and using the fact that

$$\int_t^\infty \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r} \geq c_3\frac{\omega(B(x,r))}{r^{N+2-\alpha}},$$

we get

$$\int_0^\infty t\left(\int_t^\infty \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \geq c_3\int_0^\infty r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr.$$

Thus, we get (4.4.29). Likewise, we also obtain (4.4.30). ∎

We have comparisons of $\text{Cap}_{\mathcal{H}_\alpha,p}, \text{Cap}_{\mathcal{G}_\alpha,p}, \text{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}, \text{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}$.

Corollary 4.4.20 *Let $p>1$ and $\frac{2}{p}<\alpha<N+\frac{2}{p}$. There exists a constant C depending on N,q,α such that for any compact $K\subset\mathbb{R}^N$*

$$C^{-1}\,Cap_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K) \leq Cap_{\mathcal{H}_\alpha,p}(K\times\{0\}) \leq C\,Cap_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K) \tag{4.4.32}$$

and

$$C^{-1}\,Cap_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K) \leq Cap_{\mathcal{G}_\alpha,p}(K\times\{0\}) \leq C\,Cap_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K) \tag{4.4.33}$$

Proof. By [2, Chapter 2], we have

$$\begin{aligned}
\text{Cap}_{\mathcal{H}_\alpha,p}(K\times\{0\})^{1/p} &= \sup\{\mu(K\times\{0\}) : \mu\in\mathfrak{M}^+(K\times\{0\}), ||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}\leq 1\} \\
&= \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^{p'}(\mathbb{R}^{N+1})}\leq 1\},\\
\text{Cap}_{\mathcal{G}_\alpha,p}(K\times\{0\})^{1/p} &= \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{G}}_\alpha[\omega\otimes\delta_0]||_{L^{p'}(\mathbb{R}^{N+1})}\leq 1\},\\
\text{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K)^{1/p} &= \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\mathbf{I}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\leq 1\},\\
\text{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K)^{1/p} &= \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\mathbf{G}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\leq 1\}.
\end{aligned}$$

Therefore, thanks to Proposition (4.4.19) we get the results. ∎

Corollary 4.4.21 *Let $p>1$ and k be a positive integer such that $2k<N+2/p$. There exists a constant C depending on N,k,p such that for any compact set $K\subset\mathbb{R}^N$*

$$C^{-1}\,Cap_{\mathbf{G}_{2k-\frac{2}{p}},p}(K) \leq Cap_{2k,k,p}(K\times\{0\}) \leq C\,Cap_{\mathbf{G}_{2k-\frac{2}{p}},p}(K). \tag{4.4.34}$$

We also have comparisons of $\text{Cap}_{\mathcal{G}_\alpha,p}, \text{Cap}_{\mathbf{G}_\alpha,p}$.

Proposition 4.4.22 *Let $0 < \alpha < N$, $p > 1$. For $a > 0$ there exists a constant C depending on N, α, p, a such that for any compact $K \subset \mathbb{R}^N$,*

$$C^{-1}\,Cap_{\mathbf{G}_\alpha,p}(K) \le Cap_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \le C\,Cap_{\mathbf{G}_\alpha,p}(K).$$

Proof. By [2], we have

$$\mathrm{Cap}_{\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}},p}(K) \le c_1 \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_1 = c_1(N,\alpha,p,a) > 0$. So, we can find $f \in L^p_+(\mathbb{R}^N)$ such that $\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f \ge \chi_K$ and

$$\int_{\mathbb{R}^N} |f|^p dx \le 2c_1 \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

Note that $(E_\alpha^{\sqrt{a}} * \tilde{f})(x,t) \ge c_2(\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f)(x,t)$ for all $(x,t) \in \mathbb{R}^N \times [-a,a]$ where $\tilde{f}(x,t) = f(x)\chi_{[-2a,2a]}(t)$ and constant $c_2 = c_2(N,\alpha,p)$. So,

$$\begin{aligned}\mathrm{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a,a]) &\le c_2^{-p} \int_{\mathbb{R}^{N+1}} |\tilde{f}|^p dxdt \\ &= 4c_2^{-p} a \int_{\mathbb{R}^N} |f|^p dx.\end{aligned}$$

By Corollary 4.4.10, there is $c_1 = c_1(N,\alpha,p,a) > 0$ such that

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \le c_1 \mathrm{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a,a]).$$

Thus, we get

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \le c_3 \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_3 = c_3(N,\alpha,p,a)$.
Finally, we prove other one. It is easy to see that

$$||\mathbb{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \le c_4 ||\mathbf{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega]||_{L^{p'}(\mathbb{R}^N)} \quad \forall\, \omega \in \mathfrak{M}^+(\mathbb{R}^N),$$

for some $c_4 = c_4(N,\alpha,p)$, which implies

$$||\mathcal{G}_\alpha[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \le c_5 ||\mathbf{G}_\alpha[\omega]||_{L^{p'}(\mathbb{R}^N)} \quad \forall\, \omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$$

for some $c_4 = c_4(N,\alpha,p,a)$.
It follows,

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \ge c_6 \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K),$$

for some $c_6 = c_6(N,\alpha,p,a)$. ■

The following proposition is useful for proving that many operators of classical analysis are bounded in the space the space of functions f such that

$$\int_K |f|^p dxdt \le C\mathrm{Cap}(K)$$

for every compact set $K \subset \mathbb{R}^{N+1}$, $(1 < p < \infty)$, if they are bounded in $L^q(\mathbb{R}^{N+1}, dw)$ with $w \in A_\infty$.

Proposition 4.4.23 *Let* $0 < R \leq \infty$, $1 < p \leq \alpha^{-1}(N+2)$, $0 < \delta < \alpha$ *and* $f, g \in L^1_{loc}(\mathbb{R}^{N+1})$. *Suppose that*

1. *There exists a positive constant* C_1 *such that*

$$\int_K |f| dxdt \leq C_1 Cap_{E_\alpha^{R,\delta},p}(K) \text{ for any compact sets } K \subset \mathbb{R}^{N+1}. \tag{4.4.35}$$

2. *For all weights* $w \in A_1$,

$$\int_{\mathbb{R}^{N+1}} |g| w dxdt \leq C_2 \int_{\mathbb{R}^{N+1}} |f| w dxdt, \tag{4.4.36}$$

where the constant C_2 *depends only on* N *and* $[w]_{A_1}$.

Then,

$$\int_K |g| dxdt \leq C_3 Cap_{E_\alpha^{R,\delta},p}(K) \text{ for any compact set } K \subset \mathbb{R}^{N+1}, \tag{4.4.37}$$

where the constant C_3 *depends only on* N, α, p, δ *and* C_1, C_2.

The capacity is mentioned in the Proposition (4.4.23), that is $(E_\alpha^{R,\delta}, p)$-capacity defined by

$$\text{Cap}_{E_\alpha^{R,\delta},p}(E) = \inf \left\{ \int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), E_\alpha^{R,\delta} * f \geq \chi_E \right\},$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$, where $0 < R \leq \infty$, $0 < \delta < \alpha < N+2$,

$$E_\alpha^{R,\delta}(x,t) = \max\{|x|, \sqrt{2|t|}\}^{-(N+2-\alpha)} \min\left\{1, \left(\frac{\max\{|x|, \sqrt{2|t|}\}}{R}\right)^{-\delta}\right\}.$$

Remark 4.4.24 *For* $0 < \alpha q < N+2$, *the inequality* (4.4.10) *in Proposition 4.4.4 implies*

$$\left(\int_{\mathbb{R}^{N+1}} \left(E_\alpha^{R,\delta} * f\right)^{\frac{q(N+2)}{N+2-\alpha q}} dxdt \right)^{1-\frac{\alpha q}{N+2}} \leq C \int_{\mathbb{R}^{N+1}} f^q dxdt \quad \forall f \in L^q(\mathbb{R}^{N+1}), f \geq 0. \tag{4.4.38}$$

Hence, we get the isoperimetric inequality :

$$|E|^{1-\frac{\alpha p}{N+2}} \leq C\, Cap_{E_\alpha^{R,\delta},p}(E), \tag{4.4.39}$$

for all measurable sets $E \subset \mathbb{R}^{N+1}$.

Also, we recall that a positive function $w \in L^1_{loc}(\mathbb{R}^{N+1})$ is called an A_1 weight, if the quality

$$[w]_{A_1} := \sup \left(\left(\fint_Q w dyds \right) \underset{(x,t)\in Q}{ess\sup} \frac{1}{w(x,t)} \right) < \infty,$$

where the supremum is taken over all cylinder $Q = \tilde{Q}_R(x,t) \subset \mathbb{R}^{N+1}$. The constant $[w]_{A_1}$ is called the A_1 constant of w.

To prove the Proposition (4.4.23), we need to introduce the $(R,\delta)-$Wolff parabolic potential,

$$\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t) = \int_0^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho} \quad \text{for any } (x,t) \in \mathbb{R}^{N+1},$$

where $p > 1$, $0 < \alpha p \leq N+2$, $0 < \delta < \alpha p'$ and $R \in (0,\infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$.
It is easy to see that

$$\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t) \leq C \sup_{(y,s)\in \mathrm{supp}\mu} \mathbb{W}_{\alpha,p}^{R,\delta}[\mu](y,s). \tag{4.4.40}$$

for some a constant $C = C(N,\alpha,p,\delta) > 0$.

Remark 4.4.25 *We easily verify that the Theorem 4.4.1 also holds for* $\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu]$ *and* $\mathbb{M}_{\alpha p}^{R,\delta,R_1}[\mu]$ *:*

$$\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu](x,t) = \int_0^{R_1} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho},$$

$$\mathbb{M}_{\alpha,p}^{R,\delta/(p-1),R_1}[\mu](x,t) = \sup_{0<\rho<R_1} \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta(p-1)}\right\} \right) \quad \textit{for any } (x,t) \in \mathbb{R}^{N+1},$$

where $0 < \delta < \alpha p'$, $1 < p < \alpha^{-1}(N+2)$ *and* $R_1 > R > 0$*. This means, for* $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$*, there exist constants* $C_1, C_2 > 0$ *and* $\varepsilon_0 \in (0,1)$ *depending on* $N, \alpha, p, \delta, [w]_{A_\infty}$ *such that for any* $\lambda > 0$ *and* $\varepsilon \in (0,\varepsilon_0)$

$$w(\{\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^{R,\delta(p-1),R_1}[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\}) \leq C_1 \exp(-C_2\varepsilon^{-1}) w(\{\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu] > \lambda\}), \tag{4.4.41}$$

where $a = 2 + 3^{\frac{N+2-\alpha p+\delta(p-1)}{p-1}}$.
Therefore, for $q > p-1$

$$||\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \leq C_3 ||(\mathbb{M}_{\alpha p}^{R,\delta(p-1),R_1}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)},$$

where $C_3 = C_3(N,\alpha,p,\delta,q)$*. Letting* $R_1 \to \infty$*, we get*

$$||\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \leq C_3 ||(\mathbb{M}_{\alpha p}^{R,\delta(p-1)}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)}, \tag{4.4.42}$$

where $\mathbb{M}_{\alpha p}^{R,\delta(p-1)}[\mu] := \mathbb{M}_{\alpha p}^{R,\delta(p-1),\infty}[\mu]$.

We will need the following three Lemmas to prove the Proposition (4.4.23).

Lemma 4.4.26 *Let* $0 < p \leq \alpha^{-1}(N+2)$ *and* $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$*. There exists a constant* c *depending on* δ *such that for each* $\tilde{Q}_r = \tilde{Q}_r(x,t)$

$$\fint_{\tilde{Q}_r} (\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](y,s))^\beta dyds \leq c(\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t))^\beta. \tag{4.4.43}$$

Proof. We set

$$U^r_{\alpha,p}[\mu](y,s) = \int_r^\infty \left(\frac{|\mu|(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\}\frac{d\rho}{\rho} \quad \text{and}$$

$$L^r_{\alpha,p}[\mu](y,s) = \int_0^r \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\}\frac{d\rho}{\rho}.$$

Thus,

$$\fint_{\tilde{Q}_r} (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](y,s))^\delta dyds \le c_1 \fint_{\tilde{Q}_r} (U^r_{\alpha,p}[\mu](y,s))^\delta dyds + c_1 \fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\delta dyds.$$

Since for each $(y,s) \in \tilde{Q}_r$ and $\rho \ge r$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2\rho}(x,t)$, thus for each $(y,s) \in \tilde{Q}_r$,

$$\begin{aligned} U^r_{\alpha,p}[\mu](y,s) &\le \int_r^\infty \left(\frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \left(\max\{1,\frac{\rho}{R}\}\right)^{-\delta}\frac{d\rho}{\rho} \\ &\le c_2 \mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t), \end{aligned}$$

which implies

$$\fint_{\tilde{Q}_r} (U^r_{\alpha,p}[\mu](y,s))^\delta dyds \le c_2 (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\delta.$$

Since for each $(y,s) \in \tilde{Q}_r$ and $\rho \le r$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2r}(x,t)$ thus, $L^r_{\alpha,p}[\mu] = L^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] \le \mathbb{W}^{R,\delta}_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$ in $\tilde{Q}_r(x,t)$. We now consider two cases.
Case 1 : $r \le R$. We have for $a > 0$,

$$\begin{aligned} \fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\le \fint_{\tilde{Q}_r} (\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds \\ &= \frac{1}{|\tilde{Q}_r|}\beta \int_0^\infty \lambda^{\beta-1} |\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] > \lambda\} \cap \tilde{Q}_r| d\lambda \\ &\le a^\beta + c_2 r^{-N-2} \int_a^\infty \lambda^{\beta-1} |\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] > \lambda\}| d\lambda. \end{aligned}$$

If $\alpha p = N+2$, we use (4.4.11) in Remark 4.4.5 with $\varepsilon = \frac{\alpha p}{\beta}$ and take $a = (\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}$

$$\begin{aligned} \fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\le a^\beta + c_3 r^{-N-2} \int_a^\infty \lambda^{\beta-1} \left(\frac{(\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} r^{\alpha p} d\lambda \\ &\le c_4 (\mu(\tilde{Q}_{2r}(x,t)))^{\frac{\beta}{p-1}} \\ &\le c_5 (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \end{aligned}$$

If $\alpha p < N+2$, we use (4.4.8) in Proposition 4.4.4 and take $a = \mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}$, we get

$$\begin{aligned} \fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\le c_6 \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}\right)^\beta \\ &\le c_7 (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta. \end{aligned}$$

Case 2 : $r \geq R$. As above case, we have

$$\fint_{\tilde{Q}_r} (\mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds \leq c_6 \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}}\right)^\beta.$$

Since $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] \leq R^\delta \mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$, thus

$$\begin{aligned}\fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds &\leq c_6 \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}} R^\delta\right)^\beta \\ &\leq c_5(\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta.\end{aligned}$$

Therefore, we get (4.4.43). The proof completes. ∎

Remark 4.4.27 *It is easy to see that the inequality* (4.4.43) *does not true for* $\mathbb{W}^R_{\alpha,p}[\delta_{(0,0)}]$ *where* $\delta_{(0,0)}$ *is the Dirac mass at* $(x,t) = (0,0)$.

Remark 4.4.28 *From Lemma* (4.4.26)*, we have, if there exists* $(x_0,t_0) \in \mathbb{R}^{N+1}$ *such that* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x_0,t_0) < \infty$ *then* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \in L^\beta_{loc}(\mathbb{R}^{N+1})$ *for any* $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$.

Lemma 4.4.29 *Let* $R \in (0,\infty]$, $1 < p \leq \alpha^{-1}(N+2)$ *and* $0 < \delta < \alpha p'$. *Assume that* $\alpha p < N+2$ *if* $R = \infty$. *Then, for any compact set* $K \subset \mathbb{R}^{N+1}$ *there exists a* $\mu \in \mathfrak{M}^+(K)$, *called a capacitary measure of* K *such that*

$$C_1^{-1} Cap_{E^{R,\delta/p'}_\alpha,p}(K) \leq \mu(K) \leq C_1 Cap_{E^{R,\delta/p'}_\alpha,p}(K)$$

and $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t) \geq C_2$ *a.e in* K *and* $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \leq C_3$ *a.e in* $\mathbb{R}^{N+1}$ *for some constants* $C_i = C_i(N,\alpha,p)$, $i = 1,2,3$.

Proof. We consider a measure ν on $M = \mathbb{R}^{N+1} \times \mathbb{Z}$ as follows

$$\nu = m \otimes \sum_{n=-\infty}^{\infty} \delta_n,$$

where m is Lebesgue measure, and δ_n denotes unit mass at n. Thus, $f \in L^p(M,d\nu)$, means $f = \{f_n\}_{-\infty}^\infty$, with

$$||f||^p_{L^p(M,d\nu)} = \sum_{n=-\infty}^{\infty} ||f_n||^p_{L^p(\mathbb{R}^{N+1})}.$$

Let $n_R \in \mathbb{Z} \cup \{+\infty\}$ such that $2^{-n_R} \leq R < 2^{-n_R+1}$ if $R < +\infty$ and $n_R - \infty$ if $R = +\infty$. We define a kernel $\mathbb{P}_\alpha$ in $\mathbb{R}^{N+1} \times M = \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{Z}$ by

$$\mathbb{P}_\alpha(x,t,x',t',n) = \min\{1, 2^{(n-n_R)\delta/p'}\} 2^{n(N+2-\alpha)} \chi_{\tilde{Q}_{2^{-n}}}(x-x',t-t').$$

If f is $\nu-$measurable and nonnegative and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, the corresponding potentials $\mathcal{P}_\alpha f$, $\overset{\vee}{\mathcal{P}}_\alpha \mu$ and $V^\mu_{\mathbb{P}_\alpha,p}$ are everywhere well defined and given by

$$\begin{aligned}
(\mathcal{P}_\alpha f)(x,t) &= \int_M \mathbb{P}_\alpha(x,t,x',t',n) f(x',t',n) d\nu(x',t',n) \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta/p'}\} 2^{n(N+2-\alpha)} (\chi_{\tilde{Q}_{2^{-n}}} * f_n)(x,t), \\
(\overset{\vee}{\mathcal{P}}_\alpha \mu)(x',t',n) &= \int_{\mathbb{R}^{N+1}} \mathbb{P}_\alpha(x,t,x',t',n) d\mu(x,t) \\
&= \min\{1, 2^{(n-n_R)\delta/p'}\} 2^{n(N+2-\alpha)} (\chi_{\tilde{Q}_{2^{-n}}} * \mu)(x',t'), \\
V^\mu_{P_\alpha,p}(x,t) &= (\mathcal{P}_\alpha (\overset{\vee}{\mathcal{P}}_\alpha \mu)^{p'-1})(x,t) \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta}\} 2^{np'(N+2-\alpha)} \left(\chi_{\tilde{Q}_{2^{-n}}} * \left(\chi_{\tilde{Q}_{2^{-n}}} * \mu \right)^{p'-1} \right)(x,t),
\end{aligned}$$

for any $(x,t,x',t',n) \in \mathbb{R}^{N+1} \times M$.
Since for all $(x,t) \in \mathbb{R}^{N+1}$,

$$\begin{aligned}
|\tilde{Q}_1| 2^{-(n+1)(N+2)} (\mu(\tilde{Q}_{2^{-n-1}}(x,t)))^{p'-1} &\le \left(\chi_{\tilde{Q}_{2^{-n}}} * \left(\chi_{\tilde{Q}_{2^{-n}}} * \mu \right)^{p'-1} \right)(x,t) \\
&\le |\tilde{Q}_1| 2^{-n(N+2)} (\mu(\tilde{Q}_{2^{-n+1}}(x,t)))^{p'-1},
\end{aligned}$$

thus,

$$c_1^{-1} V^\mu_{\mathbb{P}_\alpha,p} \le \mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \le c_1 V^\mu_{\mathbb{P}_\alpha,p}, \tag{4.4.44}$$

for some a positive constant c_1.
We now define the L^p-capacity with $1 < p < \infty$

$$\mathrm{Cap}_{\mathbb{P}_\alpha,p}(E) = \inf\{||f||^p_{L^p(M,d\nu)} : f \in L^p_+(M,d\nu), \mathcal{P}_\alpha f \ge \chi_E\}.$$

for any Borel set $E \subset \mathbb{R}^{N+1}$. By [2, Theorem 2.5.1], for any compact set $K \subset \mathbb{R}^{N+1}$

$$\mathrm{Cap}_{\mathbb{P}_\alpha,p}(K)^{1/p} = \sup\{\mu(K) : \mu \in \mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{P}}_\alpha \mu||_{L^{p'}(M,d\nu)} \le 1\}.$$

By [2, Theorem 2.5.6], for any compact set K in $\mathbb{R}^{N+1}$, there exists $\mu \in \mathfrak{M}^+(K)$, called a capacitary measure for K, such that $V^\mu_{\mathbb{P}_\alpha,p} \ge 1$ $\mathrm{Cap}_{\mathbb{P}_\alpha,p}-$ q.e. in K, $V^\mu_{P_\alpha,p} \le 1$ a.e in supp(μ) and $\mu(K) = \mathrm{Cap}_{\mathbb{P}_\alpha,p}(K)$. Thanks to (4.4.44) and (4.4.40), we have $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \ge c_1^{-1}$ $\mathrm{Cap}_{\mathbb{P}_\alpha,p}-$ q.e. in K, $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \le c_2$ a.e in $\mathbb{R}^{N+1}$ and $\mu(K) = \mathrm{Cap}_{\mathbb{P}_\alpha,p}(K)$.
On the other hand,

$$\begin{aligned}
||\overset{\vee}{\mathcal{P}}_\alpha \mu||^{p'}_{L^{p'}(M,d\nu)} &= \sum_{n=-\infty}^{\infty} ||\min\{1, 2^{(n-n_R)\delta/p'}\} 2^{n(N+2-\alpha)} \chi_{\tilde{Q}_{2^{-n}}} * \mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta}\} 2^{np'(N+2-\alpha)} \int_{\mathbb{R}^{N+1}} (\chi_{\tilde{Q}_{2^{-n}}} * \mu)^{p'} dxdt,
\end{aligned}$$

this quantity is equivalent to

$$\int_{\mathbb{R}^{N+1}}\int_0^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}}\right)^{p'}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}dxdt.$$

So, thanks to (4.4.42) in Remark 4.4.25, we obtain

$$c_2^{-1}||E_\alpha^{R,\delta/p'}*\mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})}\le||\overset{\vee}{\mathcal{P}}_\alpha\mu||^{p'}_{L^{p'}(M,d\nu)}\le c_2||E_\alpha^{R,\delta/p'}*\mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})}.$$

for $c_2=c_2(N,p,\alpha,\delta)$. It follows that two capacities $\mathrm{Cap}_{\mathbb{P}_\alpha,p}$and $\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}$ are equivalent. Therefore, we obtain the desired results. ■

Lemma 4.4.30 *Let $R\in(0,\infty]$, $1<p\le\alpha^{-1}(N+2)$ and $0<\delta<\alpha p'$. Assume that $\alpha p<N+2$ if $R=\infty$. Then there exists $C=C(N,\alpha,p,\delta)$ such that for any $\mu\in\mathfrak{M}_b^+(\mathbb{R}^{N+1})$*

$$Cap_{E_\alpha^{R,\delta/p'},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\})\le C\lambda^{-p+1}\mu(\mathbb{R}^{N+1})\ \ \forall\,\lambda>0. \tag{4.4.45}$$

In particular, $\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]<\infty$ $Cap_{E_\alpha^{R,\delta/p'},p}-q.e.$ in $\mathbb{R}^{N+1}$.

Proof. By Lemma 4.4.29, there is a capacitary measure σ for a compact subset K of $\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\}$ such that $\mathbb{W}_{\alpha,p}^{R,\delta}[\sigma](x,t)\le c_1$ on suppσ and $\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}(K)\approx\sigma(K)$ where $c_1=c_1(N,\alpha,p,\delta)$.
Set $\mathbb{M}[\mu,\sigma](x,t)=\sup_{\rho>0}\frac{\mu(\tilde{Q}_\rho(x,t))}{\sigma(\tilde{Q}_{3\rho}(x,t))}$ for any $(x,t)\in$ suppσ. Then, for any $(x,t)\in$ suppσ

$$\begin{aligned}\lambda<\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t)&\le(\mathbb{M}[\mu,\sigma](x,t))^{\frac{1}{p-1}}\int_0^\infty\left(\frac{\sigma(\tilde{Q}_{3\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}\\&\le c_2\left(\mathbb{M}[\mu,\sigma](x,t)\right)^{\frac{1}{p-1}}.\end{aligned}$$

Thus, for any $\lambda>0$, supp$\sigma\subset\{c_2\left(\mathbb{M}[\mu,\sigma]\right)^{\frac{1}{p-1}}>\lambda\}=\{\mathbb{M}[\mu,\sigma]>\left(\frac{\lambda}{c_2}\right)^{p-1}\}$. By Vitali Covering Lemma one can cover suppσ with a union of $\tilde{Q}_{3\rho_i}(x_i,t_i)$ for $i=1,...,m(K)$ so that $\tilde{Q}_{\rho_i}(x_i,t_i)$ are disjoint and $\sigma(\tilde{Q}_{3\rho_i}(x_i,t_i))<(\lambda/c_2)^{-p+1}\mu(\tilde{Q}_{\rho_i}(x_i,t_i))$. It follows that

$$\begin{aligned}\mathrm{Cap}_{E_\alpha^R,p}(K)&\le c_3\sum_{i=1}^{m(K)}\sigma(\tilde{Q}_{3\rho_i}(x_i,t_i))\\&\le c_3c_2^{p-1}\lambda^{-p+1}\sum_{i=1}^{m(K)}\mu(\tilde{Q}_{\rho_i}(x_i,t_i))\\&\le c_3c_2^{p-1}\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).\end{aligned}$$

So, for all compact subset K of $\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\}$,

$$\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}(K)\le c_1c_2^{p-1}\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).$$

Therefore we obtain (4.4.45). ■

Remark 4.4.31 *Let $0<\delta<\alpha<N+2$ and $\delta\le 1$. From the following inequality*

$$\begin{aligned}&|\max\{|x_1-z|,\sqrt{2|t_1-s|}\}^{-N-2+\alpha}-\max\{|x_2-z|,\sqrt{2|t_2-s|}\}^{-N-2+\alpha}|\\ &\le c_1\left(\max\{|x_1-z|,\sqrt{2|t_1-s|}\}^{-N-2+\alpha-\delta}+\max\{|x_2-z|,\sqrt{2|t_2-s|}\}^{-N-2+\alpha-\delta}\right)\\ &\quad\times\left(|x_1-x_2|+|t_1-t_2|^{1/2}\right)^{\delta},\end{aligned}$$

for all $(x_1,t_1),(x_2,t_2),(z,s)\in\mathbb{R}^{N+1}$, where c_1 is a constant depending on N,α,δ. Thus, for $\mu\in\mathfrak{M}_b^+(\mathbb{R}^{N+1})$

$$|\mathbb{I}_\alpha[\mu](x_1,t_1)-\mathbb{I}_\alpha[\mu](x_2,t_2)|\le c_2\left(\mathbb{I}_{\alpha-\delta}[\mu](x_1,t_1)+\mathbb{I}_{\alpha-\delta}[\mu](x_2,t_2)\right)\left(|x_1-x_2|+|t_1-t_2|^{1/2}\right)^{\delta},$$

for all $(x_1,t_1),(x_2,t_2)\in\mathbb{R}^{N+1}$ and $c_2=c_1\frac{N+2-\alpha+\delta}{N+2-\alpha}$.
Consequently, for any $\mu\in\mathfrak{M}_b^+(\mathbb{R}^{N+1})$, $\mathbb{I}_\alpha[\mu]$ is $\delta-$Holder $Cap_{E_{\frac{\alpha-\delta}{2}},2}$-quasicontinuous this means, for any $\varepsilon>0$ there exists a Borel set $O_\varepsilon\subset\mathbb{R}^{N+1}$ and $c_\varepsilon>0$ such that

$$|\mathbb{I}_\alpha[\mu](x_1,t_1)-\mathbb{I}_\alpha[\mu](x_2,t_2)|\le c_\varepsilon\left(|x_1-x_2|+|t_1-t_2|^{1/2}\right)^{\delta}\quad\forall (x_1,t_1),(x_2,t_2)\in O_\varepsilon$$

and $Cap_{E_{\frac{\alpha-\delta}{2}},2}(\mathbb{R}^{N+1}\backslash O_\varepsilon)<\varepsilon$.

Now we are ready to prove Proposition 4.4.23.
Proof of Proposition 4.4.23. By Lemma 4.4.26, 4.4.29 and 4.4.30, there is the capacitary measure μ of a compact subset $K\subset\mathbb{R}^{N+1}$ such that $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]\ge c_1$ a.e in K, $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]\le c_2$ a.e in $\mathbb{R}^{N+1}$ and $\text{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda\})\le c_2\lambda^{-p+1}Cap_{E_\alpha^{R,\delta},p}(K)$ for all $\lambda>0$, $(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta\in A_1$ for any $0<\beta<\frac{(N+2)(p-1)}{N+2-\alpha p+\delta p}$. From second assumption we have

$$\int_{\mathbb{R}^{N+1}}|g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt\le C_2\int_{\mathbb{R}^{N+1}}|f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt.$$

Thus

$$\begin{aligned}\int_K|g|dxdt&\le c_1^{-\delta}\int_{\mathbb{R}^{N+1}}|g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt\\ &\le c_3\int_{\mathbb{R}^{N+1}}|f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt\\ &=c_3\beta\int_0^{c_1}\int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda}|f|dxdt\lambda^{\beta-1}d\lambda.\end{aligned}$$

By first assumption we get

$$\int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda}|f|dxdt\le C_1\text{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda\})\le c_4\lambda^{-p+1}\text{Cap}_{E_\alpha^{R,\delta},p}(K).$$

Therefore,

$$\int_K|g|dxdt\le c_5\delta\int_0^{c_1}\lambda^{-p+1}\text{Cap}_{E_\alpha^{R,\delta},p}(K)\lambda^{\delta-1}d\lambda=c_6\text{Cap}_{E_\alpha^{R,\delta},p}(K),$$

since one can choose $\delta>p-1$. This completes the proof of the Proposition. ■

Definition 4.4.32 *Let $s > 1$, $\alpha > 0$. We define the space $\mathfrak{M}^{\mathcal{H}_{\alpha},s}(\mathbb{R}^{N+1})$ ($\mathfrak{M}^{\mathcal{G}_{\alpha},s}(\mathbb{R}^{N+1})$ resp.) to be the set of all measure $\mu \in \mathfrak{M}(\mathbb{R}^{N+1})$ such that*

$$[\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha},s}(\mathbb{R}^{N+1})} := \sup\left\{|\mu|(K)/Cap_{\mathcal{H}_{\alpha},s}(K) : \ Cap_{\mathcal{H}_{\alpha},s}(K) > 0\right\} < \infty,$$
$$\left([\mu]_{\mathfrak{M}^{\mathcal{G}_{\alpha},s}(\mathbb{R}^{N+1})} := \sup\left\{|\mu|(K)/Cap_{\mathcal{G}_{\alpha},s}(K) : Cap_{\mathcal{G}_{\alpha},s}(K) > 0\right\} < \infty \ resp.\right)$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^{N+1}$.
For simplicity, we will write $\mathfrak{M}^{\mathcal{H}_{\alpha},s}, \mathfrak{M}^{\mathcal{G}_{\alpha},s}$ to denote $\mathfrak{M}^{\mathcal{H}_{\alpha},s}(\mathbb{R}^{N+1}), \mathfrak{M}^{\mathcal{G}_{\alpha},s}(\mathbb{R}^{N+1})$ resp.

We see that if $\alpha s \geq N+2$, $\mathfrak{M}^{\mathcal{H}_{\alpha},s}(\mathbb{R}^{N+1}) = \{0\}$, if $\alpha s < N+2$, $\mathfrak{M}^{\mathcal{H}_{\alpha},s}(\mathbb{R}^{N+1}) \subset \mathfrak{M}^{\mathcal{G}_{\alpha},s}(\mathbb{R}^{N+1})$. On the other hand, $\mathfrak{M}^{\mathcal{G}_{\alpha},s}(\mathbb{R}^{N+1}) \supset \mathfrak{M}_b(\mathbb{R}^{N+1})$ if $\alpha s > N+2$.
We now have the following two remarks :

Remark 4.4.33 *For $s > 1$, there is $C = C(N, \alpha, s) > 0$ such that*

$$[f]_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}} \leq C[|f|^s]^{1/s}_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}} \quad \text{for all function } f. \tag{4.4.46}$$

Indeed, set $a = [|f|^s]_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}}$, so for any compact set K in $\mathbb{R}^{N+1}$,

$$\int_K |f|^s dxdt \leq a\, Cap_{\mathcal{G}_{\alpha},p}(K).$$

This gives $2a\,Cap_{\mathcal{G}_{\alpha},p}(K) \geq \int_K (|f|^s + c_1 a)\,dxdt \geq c_2 a^{1-1/s}\int_K |f|dxdt$, here we used (4.4.24) in Remark 4.4.11 at the first inequality and Holder's inequality at the second one. It follows (4.4.46).

Remark 4.4.34 *Assume that $p > 1$ and $\frac{2}{p} < \alpha < N + \frac{2}{p}$. Clearly, from Corollary 4.4.20 we assert that for $\omega \in \mathfrak{M}^+(\mathbb{R}^N)$*

$$C_1^{-1}\,[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}} \leq \left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},p}} \leq C_1\,[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}\,,$$
$$C_2^{-1}\,[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}} \leq \left[\omega \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}} \leq C_2\,[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}\,,$$

for some $C_i = C_i(N, p, \alpha)$, $i = 1, 2$. Where $\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p} := \mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)$, $\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p} := \mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)$ and

$$[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)} := \sup\left\{\omega(K)/Cap_{\mathbf{I}_{\alpha-2/p},p}(K) : \ Cap_{\mathbf{I}_{\alpha-2/p},p}(K) > 0\right\},$$
$$[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)} := \sup\left\{\omega(K)/Cap_{\mathbf{G}_{\alpha-2/p},p}(K) : \ Cap_{\mathbf{G}_{\alpha-2/p},p}(K) > 0\right\},$$

where the supremum is taken all compact sets $K \subset \mathbb{R}^N$.

Clearly, Theorem 4.4.2 and Proposition 4.4.23 lead to the following result.

Proposition 4.4.35 *Let $q > p - 1$, $s > 1$ and $0 < \alpha p < N + 2$. Then the following quantities are equivalent*

$$\left[\left(\mathbb{W}^R_{\alpha,p}[\mu]\right)^q\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},s}}, \quad \left[\left(\mathbb{I}^R_{\alpha p}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},s}} \quad and \quad \left[\left(\mathbb{M}^R_{\alpha p}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},s}},$$

for every $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ and $0 < R \leq \infty$.

In the next result, we present a characterization of the following trace inequality :

$$||E_\alpha^{R,\delta} * f||_{L^p(\mathbb{R}^{N+1},d\mu)} \le C_1||f||_{L^p(\mathbb{R}^{N+1})} \quad \forall f \in L^p(\mathbb{R}^{N+1}). \tag{4.4.47}$$

Theorem 4.4.36 *Let $0 < R \le \infty$,$1 < p < \alpha^{-1}(N+2)$, $0 < \delta < \alpha$ and μ be a nonnegative Radon measure on $\mathbb{R}^{N+1}$. Then the following statements are equivalent.*

1. *The trace inequality (4.4.47) holds.*

2. *There holds*

$$||E_\alpha^{R,\delta} * f||_{L^p(\mathbb{R}^{N+1},d\omega)} \le C_2||f||_{L^p(\mathbb{R}^{N+1})} \quad \forall f \in L^p(\mathbb{R}^{N+1}), \tag{4.4.48}$$

where $d\omega = (\mathbb{I}_\alpha^{R,\delta}\mu)^{p'}dxdt$.

3. *There holds*

$$||E_\alpha^{R,\delta} * f||_{L^{p,\infty}(\mathbb{R}^{N+1},d\mu)} \le C_3||f||_{L^p(\mathbb{R}^{N+1})} \quad \forall f \in L^p(\mathbb{R}^{N+1}). \tag{4.4.49}$$

4. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\mu(E) \le C_4 Cap_{E_\alpha^{R,\delta},p}(E). \tag{4.4.50}$$

5. $\mathbb{I}_\alpha^{R,\delta}[\mu] < \infty$ *a.e and*

$$\mathbb{I}_\alpha^{R,\delta}[(\mathbb{I}_\alpha^{R,\delta}[\mu])^{p'}] \le C_5\mathbb{I}_\alpha^{R,\delta}[\mu] \quad a.e. \tag{4.4.51}$$

6. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_E (\mathbb{I}_\alpha^{R,\delta}[\mu])^{p'}dxdt \le C_6 Cap_{E_\alpha^{R,\delta},p}(E). \tag{4.4.52}$$

7. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha^{R,\delta}[\mu\chi_E])^{p'}dxdt \le C_7\mu(E). \tag{4.4.53}$$

8. *For every compact set $E \subset \mathbb{R}^{N+1}$,*

$$\int_E (\mathbb{I}_\alpha^{R,\delta}[\mu\chi_E])^{p'}dxdt \le C_8\mu(E). \tag{4.4.54}$$

We can find a simple sufficient condition on μ so that trace inequality (4.4.47) is satisfied from the isoperimetric inequality (4.4.39).

Proof of Theorem 4.4.36. As in [80] we can show that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6 \Leftrightarrow 7$ and $7 \Rightarrow 8, 5 \Rightarrow 2$. Thus, it is enough to show that $8. \Rightarrow 5$. First, we need to show that

$$\left(\int_r^\infty \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}\right)^{p'-1} \le c_1 r^{-\alpha}\left(\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)^{-1} \tag{4.4.55}$$

We have for any $(y,s) \in \tilde{Q}_r(x,t)$

$$\begin{aligned}\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_r(x,t)}](y,s) &= \int_0^\infty \frac{\mu(\tilde{Q}_r(x,t)\cap\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha}}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}\\ &\geq \int_{2r}^{4r} \frac{\mu(\tilde{Q}_r(x,t)\cap\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha}}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}\\ &\geq c_2\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}.\end{aligned}$$

In (4.4.54), we take $E = \tilde{Q}_r(x,t)$

$$\begin{aligned}c\mu(\tilde{Q}_r(x,t)) &\geq \int_{\tilde{Q}_r(x,t)} (\mathbb{I}_\alpha[\mu\chi_{\tilde{Q}_r(x,t)}])^{p'}\\ &\geq c_2^{p'}\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)^{p'}|\tilde{Q}_r(x,t)|.\end{aligned}$$

So $\mu(\tilde{Q}_r(x,t)) \leq c_3 r^{N+2-\alpha p}\left(\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)^{-p}$ which implies (4.4.55).
Next we set

$$L_r[\mu](x,t) = \int_r^{+\infty}\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho},$$

$$U_r[\mu](x,t) = \int_0^{r}\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho},$$

and

$$d\omega = (I_\alpha\mu)^{p'}dxdt, \quad d\sigma_{1,r} = (L_r[\mu])^{p'}dxdt, \quad d\sigma_{2,r} = (U_r[\mu])^{p'}dxdt.$$

We have $d\omega \leq 2^{p'-1}\left(d\sigma_{1,r}+d\sigma_{2,r}\right)$. To prove (4.4.51) we need to show that

$$\int_0^\infty \frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\frac{dr}{r} \leq c_4\mathbb{I}_\alpha^{R,\delta}[\mu](x,t), \tag{4.4.56}$$

$$\int_0^\infty \frac{\sigma_{2,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\frac{dr}{r} \leq c_5\mathbb{I}_\alpha^{R,\delta}[\mu](x,t). \tag{4.4.57}$$

Since, for all $r>0$, $0<\rho<r$ and $(y,s)\in\tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s)\subset\tilde{Q}_{2r}(x,t)$. So,

$$\sigma_{2,r}(\tilde{Q}_r(x,t)) = \int_{\tilde{Q}_r(x,t)}(U_r[\mu](y,s))^{p'}dyds = \int_{\tilde{Q}_r(x,t)}\left(U_r[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds.$$

Thus, from (4.4.54) we get

$$\begin{aligned}\sigma_{2,r}(\tilde{Q}_r(x,t)) &\leq \int_{\tilde{Q}_{2r}(x,t)}\left(U_r[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds\\ &\leq \int_{\tilde{Q}_{2r}(x,t)}\left(\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s)\right)^{p'}dyds\\ &\leq c_6\mu(\tilde{Q}_{2r}(x,t)).\end{aligned}$$

Therefore, (4.4.57) follows.
Since, for all $r > 0$, $\rho \geq r$ and $(y,s) \in \tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2\rho}(x,t)$. So, for all $(y,s) \in \tilde{Q}_r(x,t)$ we have

$$\begin{aligned} L_r[\mu](y,s) &\leq \int_r^{+\infty} \frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha}} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho} \\ &\leq c_7 L_r[\mu](x,t). \end{aligned}$$

Hence,

$$\begin{aligned} \sigma_{1,r}(\tilde{Q}_r(x,t)) &= \int_{\tilde{Q}_r(x,t)} \left(L_r[\mu](y,s)\right)^{p'} dyds \\ &\leq c_8 r^{N+2} \left(L_r[\mu](x,t)\right)^{p'}. \end{aligned}$$

Since $r^{\alpha-1}\min\{1, \left(\frac{r}{R}\right)^{-\delta}\} \leq \frac{1}{\alpha-\delta}\frac{d}{dr}\left(r^\alpha \min\{1, \left(\frac{r}{R}\right)^{-\delta}\}\right)$, we deduce that

$$\begin{aligned} \int_0^\infty \frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\{1, \left(\frac{r}{R}\right)^{-\delta}\} \frac{dr}{r} &\leq c_7 \int_0^\infty r^{\alpha-1} \left(L_r[\mu](x,t)\right)^{p'} \min\{1, \left(\frac{r}{R}\right)^{-\delta}\} dr \\ &\leq \frac{c_7}{\alpha-\delta} \int_0^\infty \frac{d}{dr}\left(r^\alpha \min\{1, \left(\frac{r}{R}\right)^{-\delta}\}\right) \left(L_r[\mu](x,t)\right)^{p'} dr \\ &\leq c_8 \int_0^\infty r^\alpha \left(L_r[\mu](x,t)\right)^{p'-1} \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\{1, \left(\frac{r}{R}\right)^{-\delta}\}^2 \frac{dr}{r}. \end{aligned}$$

Therefore, we get (4.4.56) from (4.4.55). This completes the proof of Theorem. ■

Remark 4.4.37 *It is easy to assert that if **8.** holds then for any* $0 < \beta < N+2$

$$\mathbb{I}_\beta\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right] \leq C\mathbb{I}_\beta[\mu], \tag{4.4.58}$$

for some $C = C(N,\alpha,\beta,\delta,p) > 0$.

Corollary 4.4.38 *Let* $p > 1, \alpha > 0$ *such that* $0 < \alpha p < N+2$. *There holds*

$$C_1^{-1}\left[\mu\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}}^{p'} \leq \left[\left(\mathbb{I}_\alpha[\mu]\right)^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq C_1\left[\mu\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}}^{p'} \tag{4.4.59}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. *Furthermore,*

$$[\varphi_n * \mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \leq C_2\left[\mu\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \tag{4.4.60}$$

for $n \in \mathbb{N}$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ *where* $\{\varphi_n\}$ *is a sequence of mollifiers in* $\mathbb{R}^{N+1}$. *Here* $C_i = C_i(N,p,\alpha)$, $i = 1,2$.

Proof. For $R = \infty$ we have $\mathbb{I}_\alpha^{R,\delta}[\mu] = \mathbb{I}_\alpha[\mu]$ and $E_\alpha^{R,\delta} = E_\alpha$. Thus, by (4.4.20) in Corollary 4.4.10 and Theorem 4.4.36 we get for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\mu(E) \leq c_1 \mathrm{Cap}_{\mathcal{H}_{\alpha,p}}(E)$$

if and only if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E (\mathbb{I}_\alpha[\mu])^{p'} dxdt \le c_2 \mathrm{Cap}_{\mathcal{H}_\alpha,p}(E).$$

It follows (4.4.59).
Since $\mathbb{I}_\alpha[\varphi_n * \mu] = \varphi_n * \mathbb{I}_\alpha[\mu] \le \mathbb{M}\left(\mathbb{I}_\alpha[\mu]\right)$ and $\mathbb{M}$ is bounded in $L^{p'}(\mathbb{R}^{N+1}, dw)$ with $w \in A_{p'}$ yield

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\varphi_n * \mu])^{p'} dw \le c_3([w]_{A_{p'}}) \int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\mu])^{p'} dw.$$

Thanks to Proposition 4.4.23 we have

$$\left[(\mathbb{I}_\alpha[\varphi_n * \mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}} \le c_4 \left[(\mathbb{I}_\alpha[\mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}},$$

which implies (4.4.60). ■

Corollary 4.4.39 *Let $p > 1$, $\alpha > 0$ with $0 < \alpha p \le N + 2$, $0 < \delta < \alpha$ and $R, d > 0$. There holds*

$$\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \le C_1(d/R, R) \, [\mu]^{p'}_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \tag{4.4.61}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with diam(supp(μ)) $\le d$. Furthermore,

$$[\varphi_n * \mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \le C_2(d) \, [\mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \tag{4.4.62}$$

for $n \in \mathbb{N}$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with diam(supp(μ)) $\le d$ where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Proof. It is easy to see that

$$(c_1(d/R))^{-1} ||E_\alpha^R[\mu]||_{L^{p'}(\mathbb{R}^{N+1})} \le ||E_\alpha^{R,\delta} * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \le c_1(d/R) ||E_\alpha^R[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}$$

for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with $\mathrm{diam}(\mathrm{supp}(\mu)) \le d$, thus two quantities $\mathrm{Cap}_{E_\alpha^{R,\delta},p}(E)$ and $\mathrm{Cap}_{E_\alpha^R,p}(E)$ are equivalent for every compact set $E \subset \mathbb{R}^{N+1}$, $\mathrm{diam}(E) \le d$ where equivalent constants depend only on N, p, α and d/R. Therefore, by Corollary 4.4.10 we get $\mathrm{Cap}_{E_\alpha^{R,\delta},p}(E) \approx \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E)$ for every compact set $E \subset \mathbb{R}^{N+1}$, $\mathrm{diam}(E) \le d$ where equivalent constants depend on d/R and R. Thus, by Theorem 4.4.36 and $\mathrm{diam}(\mathrm{supp}(\mu)) \le d$ we get, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\mu(E) \le c_2(d/R, R) \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E),$$

then for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E \left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'} dxdt \le c_3(d/R, R) \mathrm{Cap}_{E_\alpha^{R,\delta},p}(E) \le c_4(d/R, R) \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E).$$

It follows (4.4.61). As in the Proof of Corollary 4.4.38 we also have for $w \in A_{p'}$

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{1,\delta}[\varphi_n * \mu]\right)^{p'} dw \le c_5([w]_{A_{p'}}) \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{1,\delta}[\mu]\right)^{p'} dw.$$

Thanks to Proposition 4.4.23 and Theorem 4.4.36 we obtain (4.4.62). ■

Remark 4.4.40 *Likewise (see [71, Lemma 5.7]), we can verify that if* $\frac{2}{p} < \alpha < N + \frac{2}{p}$,

$$[\varphi_{1,n} * \omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}} \le C_1 [\omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}} \quad \text{and}$$
$$[\varphi_{1,n} * \omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}} \le C_2(d) [\omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}},$$

for $n \in \mathbb{N}$ *and* $\omega_1, \omega_2 \in \mathfrak{M}^+(\mathbb{R}^N)$ *with diam(supp(*ω_2*))* $\le d$ *where* $C_1 = C_1(N,\alpha,p)$, $C_2(d) = C_2(N,\alpha,p,d)$, $\{\varphi_{1,n}\}$ *is a sequence of standard mollifiers in* $\mathbb{R}^N$ *and* $[.]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}, [.]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}$ *was defined in Remark 4.4.34. Hence, we obtain*

$$\left[(\varphi_{1,n} * \omega_1) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},p}} \le C_3 \left[\omega_1 \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha},p}},$$
$$\left[(\varphi_{1,n} * \omega_2) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}} \le C_4(d) \left[\omega_2 \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_{\alpha},p}},$$

for $n \in \mathbb{N}$ *and* $\omega_1, \omega_2 \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, *diam(supp(*$\mu$*))* $\le d$ *where* $C_3 = C_3(N,\alpha,p)$, $C_4(d) = C_4(N,\alpha,p,d)$.

Proposition 4.4.41 *Let* $q > 1$, $0 < \alpha q < N+2$, $0 < R \le \infty, 0 < \delta < \alpha$ *and* $K > 0$. *Let* $0 \le f \in L^q_{loc}(\mathbb{R}^{N+1})$. *Let* C_4, C_5 *be constants in inequalities* (4.4.50) *and* (4.4.51) *in Theorem 4.4.36 with* $p = q'$. *Suppose that* $\{u_n\}$ *is a sequence of nonnegative measurable functions in* $\mathbb{R}^{N+1}$ *satisfying*

$$\begin{aligned} u_{n+1} &\le K\mathbb{I}_{\alpha}^{R,\delta}[u_n^q] + f \quad \forall n \in \mathbb{N} \\ u_0 &\le f \end{aligned} \tag{4.4.63}$$

Then, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E f^q dxdt \le C\, Cap_{E_{\alpha}^{R,\delta},q'}(E) \tag{4.4.64}$$

with

$$C \le C_4 \left(\frac{2^{-q+1}}{C_5(q-1)} \left(\frac{q-1}{qK2^{q-1}} \right)^q \right)^{q-1}, \tag{4.4.65}$$

then

$$u_n \le \frac{Kq2^{q-1}}{q-1} \mathbb{I}_{\alpha}^{R,\delta}[f^q] + f \quad \forall n \in \mathbb{N}. \tag{4.4.66}$$

Proof. From (4.4.50) and (4.4.51) in Theorem 4.4.36, we see that (4.4.64) implies

$$\mathbb{I}_{\alpha}^{R,\delta}[(I_{\alpha}^{R,\delta}[f^q])^q] \le \left(\frac{C}{C_4}\right)^{\frac{1}{q-1}} C_5 \mathbb{I}_{\alpha}^{R,\delta}[f^q]. \tag{4.4.67}$$

Now we prove (4.4.66) by induction. Clearly, (4.4.66) holds with $n = 0$. Next we assume that (4.4.66) holds with $n = m$. Then, by (4.4.65), (4.4.67) and (4.4.63) we have

$$\begin{aligned} u_{m+1} &\le K\mathbb{I}_{\alpha}^{R,\delta}[u_n^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1}\right)^q \mathbb{I}_{\alpha}^{R,\delta}[(\mathbb{I}_{\alpha}^{R,\delta}[f^q])^q] + K2^{q-1}\mathbb{I}_{\alpha}^{R,\delta}[f^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1}\right)^q \left(\frac{C}{C_4}\right)^{\frac{1}{q-1}} C_5\mathbb{I}_{\alpha}^{R,\delta}[f^q] + K2^{q-1}\mathbb{I}_{\alpha}^{R,\delta}[f^q] + f \\ &\le \frac{Kq2^{q-1}}{q-1}\mathbb{I}_{\alpha}^{R,\delta}[f^q] + f. \end{aligned}$$

Therefore (4.4.66) also holds true with $n = m+1$. This completes the proof of the Theorem. ■

Corollary 4.4.42 *Let $q > \frac{N+2}{N+2-\alpha}$, $\alpha > 0$ and $f \in L^q_+(\mathbb{R}^{N+1})$. There exists a constant $C > 0$ depending on N, α, q such that if for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \leq C\,Cap_{\mathcal{H}_\alpha,q'}(E)$, then $u = \mathcal{H}_\alpha[u^q] + f$ admits a positive solution $u \in L^q_{loc}(\mathbb{R}^{N+1})$.*

Proof. Consider the sequence $\{u_n\}$ of nonnegative functions defined by $u_0 = f$ and $u_{n+1} = \mathcal{H}_\alpha[u_n^q] + f \quad \forall\, n \geq 0$. It is easy to see that $u_{n+1} \leq c_1\mathbb{I}_2[u_n^q] + f \quad \forall n \geq 0$. By Proposition 4.4.41 and Corollary 4.4.38, there exists a constant $c_2 = c_2(N, \alpha, q) > 0$ such that if for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \leq c_2\mathrm{Cap}_{\mathcal{H}_\alpha,q'}(E)$ then u_n is well defined and

$$u_n \leq \frac{c_1 q 3^{q-1}}{q-1}\mathbb{I}_\alpha[f^q] + f \quad \forall n \geq 0.$$

Since $\{u_n\}$ is nondecreasing, thus thanks to the dominated convergence theorem we obtain $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a solution of $u = \mathcal{H}_\alpha[u^q]+f$ which $u \in L^q_{\mathrm{loc}}(\mathbb{R}^{N+1})$. This completes the proof of the Corollary. ■

Corollary 4.4.43 *Let $q > 1$, $\alpha > 0$, $0 < R \leq \infty, 0 < \delta < \alpha$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. The following two statements are equivalent.*

a. *for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q \leq C\,Cap_{E_\alpha^{R,\delta},q'}(E)$ for some a constant $C > 0$*

b. *There exists a function $u \in L^q_{loc}(\mathbb{R}^{N+1})$ such that $u = \mathbb{I}_\alpha^{R,\delta}[u^q] + \varepsilon f$ for some $\varepsilon > 0$.*

Proof. We will prove $b. \Rightarrow a.$ Set $d\omega(x,t) = \left(\left(\mathbb{I}_\alpha^{R,\delta}[u^q]\right)^q + \varepsilon^q f^q\right) dxdt$, thus we have $dw(x,t) \geq \left(I_\alpha^{R,\delta}[\omega]\right)^q dxdt$. Let $\mathbb{M}_\omega$ denote the centered Hardy-littlewoood maximal function which is defined for $g \in L^1_{loc}(\mathbb{R}^{N+1}, d\omega)$,

$$\mathbb{M}_\omega g(x,t) = \sup_{\rho>0} \frac{1}{\omega(\tilde{Q}_\rho(x,t))} \int_{\tilde{Q}_\rho(x,t)} |g| d\omega(x,t).$$

For $E \subset \mathbb{R}^{N+1}$ is a compact set, we have

$$\int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q dxdt \leq \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q\, d\omega(x,t).$$

Since $\mathbb{M}_\omega$ is bounded on $L^s(\mathbb{R}^{N+1}, d\omega)$ for $s > 1$ and $(\mathbb{M}_\omega\chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q \geq \left(\mathbb{I}_\alpha^{R,\delta}[\omega\chi_E]\right)^q$, thus

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{R,\delta}[\omega\chi_E]\right)^q dxdt \leq c_1\omega(E).$$

By Theorem 4.4.36, we get for any compact set $E \subset \mathbb{R}^{N+1}$

$$\omega(E) \leq c_2\mathrm{Cap}_{E_\alpha^{R,\delta},q'}(E).$$

It follows the results. ■

Remark 4.4.44 *In [64], we also use Theorem* (4.4.36) *to show existence of mild solutions to the Navier-Stokes Equations*

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}div(u \otimes u) = \mathbb{P}F & in\ \mathbb{R}^N \times (0,\infty), \\ u(0) = u_0 & in\ \mathbb{R}^N, \end{cases} \tag{4.4.68}$$

where $u, F \in \mathbb{R}^N$, $\mathbb{P} = id - \nabla\Delta^{-1}\nabla$. *is the Helmholtz Leray projection onto the vector fields of zero divergence, i.e, for* $f \in \mathbb{R}^N$, $\mathbb{P}f = f - \nabla u$ *and* $\Delta u = divf$. *Namely, there exists* $C = C(N) > 0$ *such that if* $div(u_0) = 0$ *and*

$$\int_K |D(x,t)|^2 dxdt \le C\, Cap_{\mathcal{H}_1,2}(K), \tag{4.4.69}$$

for any compact set $K \subset \mathbb{R}^{N+1}$, *where if* $(x,t) \in \mathbb{R}^N \times [0,+\infty)$,

$$D(x,t) = (e^{t\Delta}u_0)(x) + \int_0^t (e^{(t-s)\Delta}\mathbb{P}F)(x)ds,$$

and $D(x,t) = 0$ *otherwise. Then, the* (4.4.68) *has global solution* u *satisfying*

$$|u(x,t)| \le |D(x,t)| + c\mathbb{I}_1[|D|^2](x,t) \tag{4.4.70}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ *for some* $c = c(N)$.

4.5 Global point wise estimates of solutions to the parabolic equations

First, we recall Duzzar and Mingione's result [27], also see [42, 43] which involves local pointwise estimates for solutions of equations (4.2.4).

Theorem 4.5.1 *Then, there exists a constant* C *depending only* N, Λ_1, Λ_2 *such that if* $u \in L^2(0,T,H^1(\Omega)) \cap C(\Omega_T)$ *is a weak solution to* (4.2.4) *with* $\mu \in L^2(\Omega_T)$ *and* $u(0) = 0$

$$|u(x,t)| \le C \fint_{\tilde{Q}_R(x,t)} |u| dyds + C\mathbb{I}_2^{2R}[|\mu|](x,t) \tag{4.5.1}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty, T)$.
Furthermore, if A *is independent of space variable* x, (4.2.27) *is satisfied and* $\nabla u \in C(\Omega_T)$ *then*

$$|\nabla u(x,t)| \le C \fint_{\tilde{Q}_R(x,t)} |\nabla u| dyds + C\mathbb{I}_1^{2R}[|\mu|](x,t) \tag{4.5.2}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty, T)$.

Proof of Theorem 4.2.1. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega_T)$, with $\mu_0 \in \mathfrak{M}_0(\Omega_T), \mu_s \in \mathfrak{M}_s(\Omega_T)$. By Proposition 4.3.7, there exist sequences of nonnegative measures $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ and $\mu_{n,s,i}$ such that $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(\Omega_T)$ and strongly converge to some f_i, g_i, h_i in

$L^1(\Omega_T), L^2(\Omega_T, \mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively and $\mu_{n,1}, \mu_{n,2}, \mu_{n,s,1}, \mu_{n,s,2} \in C_c^\infty(\Omega_T)$ converge to $\mu^+, \mu^-, \mu_s^+, \mu_s^-$ resp. in the narrow topology with $\mu_{n,i} = \mu_{n,0,i} + \mu_{n,s,i}$, for $i = 1, 2$ and satisfying $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$ and $0 \le \mu_{n,1} \le \varphi_n * \mu^+, 0 \le \mu_{n,2} \le \varphi_n * \mu^-$, where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Let $\sigma_{1,n}, \sigma_{2,n} \in C_c^\infty(\Omega)$ be convergent to σ^+ and σ^- in the narrow topology and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ resp. such that $0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-$ where $\{\varphi_{1,n}\}$ is a sequence of standard mollifiers in $\mathbb{R}^N$. Set $\mu_n = \mu_{n,1} - \mu_{n,2}$ and $\sigma_n = \sigma_{1,n} - \sigma_{2,n}$.
Let $u_n, u_{n,1}, u_{n,2}$ be solutions of equations

$$\begin{cases} (u_n)_t - div(A(x,t,\nabla u_n)) = \mu_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n & \text{on } \Omega, \end{cases} \tag{4.5.3}$$

$$\begin{cases} (u_{n,1})_t - div(A(x,t,\nabla u_{n,1})) = \chi_{\Omega_T}\mu_{n,1} & \text{in } B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,1} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,1}(0) = \sigma_{1,n} & \text{on } B_{2T_0}(x_0), \end{cases} \tag{4.5.4}$$

$$\begin{cases} (u_{n,2})_t + div(A(x,t,-\nabla u_{n,2})) = \chi_{\Omega_T}\mu_{n,2} & \text{in } B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,2} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (0, 2T_0^2), \\ u_{n,2}(0) = \sigma_{2,n} & \text{on } B_{2T_0}(x_0), \end{cases} \tag{4.5.5}$$

where $\Omega \subset B_{T_0}(x_0)$ for $x_0 \in \Omega$.
We see that $u_{n,1}, u_{n,2} \ge 0$ in $B_{2T_0}(x_0) \times (0, 2T_0^2)$ and $-u_{n,2} \le u_n \le u_{n,1}$ in Ω_T.
Now, we estimate $u_{n,1}$. By Remark 4.3.3 and Theorem 4.3.6, a sequence $\{u_{n,1,m}\}$ of solutions to equations

$$\begin{cases} (u_{n,1,m})_t - div(A(x,t,\nabla u_{n,1,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_{n,1} & \text{in } B_{2T_0}(x_0) \times (-2T_0^2, 2T_0^2), \\ u_{n,1,m} = 0 & \text{on } \partial B_{2T_0}(x_0) \times (-2T_0^2, 2T_0^2), \\ u_{n,1,m}(-2T_0^2) = 0 & \text{on } B_{2T_0}(x_0), \end{cases} \tag{4.5.6}$$

converges to $u_{n,1}$ in $B_{2T_0}(x_0) \times (0, 2T_0^2)$, where $g_{n,m}(x,t) = \sigma_{1,n}(x) \int_{-2T_0^2}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Remark 4.3.2, we have

$$||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} \le c_1 T_0^2 A_{n,m}, \tag{4.5.7}$$

where $A_{n,m} = \mu_{n,1}(\Omega_T) + \int_{\tilde{Q}_{2T_0}(x_0,0)} \sigma_{1,n}(x)\varphi_{2,m}(t)dxdt$.
Hence, thanks to Theorem 4.5.1 we have for $(x,t) \in \Omega_T$

$$\begin{aligned} u_{n,1,m}(x,t) &\le c_8 T_0^{-N-2} ||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} + c_8 \mathbb{I}_2[\mu_{n,1}](x,t) + c_8 \mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t) \\ &\le c_9 \mathbb{I}_2[\mu_{n,1}](x,t) + c_9 \mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t). \end{aligned}$$

Since $0 \le \mu_{n,1} \le \varphi_n * \mu^+$, $\sigma_{1,n} \le \varphi_{1,n} * \sigma^+$,

$$u_{n,1,m}(x,t) \le c_9 \varphi_n * \mathbb{I}_2[\mu^+](x,t) + c_9 (\varphi_{1,n}\varphi_{2,m}) * \mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](x,t) \ \forall \ (x,t) \in \Omega_T.$$

Letting $m \to \infty$, we get

$$u_{n,1}(x,t) \le c_9 \varphi_n * \mathbb{I}_2[\mu^+](x,t) + c_9 \varphi_{1,n} * \left(\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](.,t)\right)(x) \ \forall \ (x,t) \in \Omega_T.$$

Similarly, we also get

$$u_{n,2}(x,t) \le c_9\varphi_n * \mathbb{I}_2[\mu^-](x,t) + c_9\varphi_{1,n} * \left(\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}](.,t)\right)(x) \ \forall \ (x,t) \in \Omega_T.$$

Consequently, by Proposition 4.3.5 and Theorem 4.3.6 , up to a subsequence, $\{u_n\}$ converges to a distribution solution (a renormalized solution if $\sigma \in L^1(\Omega)$) u of (4.2.4) and satisfied (4.2.7). ∎

Remark 4.5.2 *Obviously, if* $\sigma \equiv 0$ *and* $supp(\mu) \subset \overline{\Omega}\times[a,T]$, $a > 0$ *then* $u = 0$ *in* $\Omega\times(0,a)$.

Remark 4.5.3 *If* A *is independent of space variable* x, (4.2.27) *is satisfied then*

$$|\nabla u(x,t)| \le C(N,\Lambda_1,\Lambda_2,T_0/d)\mathbb{I}_1^{2T_0}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t) \tag{4.5.8}$$

for any $(x,t) \in \Omega^d \times (0,T)$ *and* $0 < d \le \frac{1}{2}\min\{\sup_{x\in\Omega} d(x,\partial\Omega), T_0^{1/2}\}$ *where* $\Omega^d = \{x \in \Omega : d(x,\partial\Omega) > d\}$. *Indeed, by Remark 4.3.3 and Theorem 4.3.6, a sequence* $\{v_{n,m}\}$ *of solutions to equations*

$$\begin{cases} (v_{n,m})_t - div(A(t,\nabla u_{n,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_n & in\ \Omega \times (-2T_0^2,T), \\ v_{n,m} = 0 \quad on\ \partial\Omega \times (-2T_0^2,T), \\ v_{n,m}(-2T_0^2) = 0 \quad on\ \Omega, \end{cases} \tag{4.5.9}$$

converges to u_n *in* $L^1(0,T,W_0^{1,1}(\Omega))$, *where* $g_{n,m}(x,t) = \sigma_n(x)\int_{-2T_0^2}^t \varphi_{2,m}(s)ds$ *and* $\{\varphi_{2,m}\}$ *is a sequence of mollifiers in* $\mathbb{R}$.
By Theorem 4.5.1, we have for any $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla v_{n,m}(x,t)| \le c_1 \fint_{\tilde{Q}_{d/2}(x,t)} |\nabla v_{n,m}|dyds + c_1\mathbb{I}_1^d[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t).$$

On the other hand, by remark 4.3.2,

$$|||\nabla v_{n,m}|||_{L^1(\Omega\times(-T_0^2,T))} \le c_2T_0(|\mu_n| + |\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T_0^2,T)).$$

Therefore, for any $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla v_{n,m}(x,t)| \le c_3\mathbb{I}_1[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t),$$

where c_3 *depends on* T_0/d.
Finally, letting $m \to \infty$ *and* $n \to \infty$ *we get for any* $(x,t) \in \Omega^d \times (0,T)$

$$|\nabla u(x,t)| \le c_3\mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t).$$

We conclude (4.5.8) *since* $\mathbb{I}_1[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}] \le c_4\mathbb{I}_1^{2T_0}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]$ *in* Ω_T.

Next, we will establish pointwise estimates from below for solutions of equations (4.2.4).

Theorem 4.5.4 *If $u \in C(Q_r(y,s)) \cap L^2(s-r^2, s, H^1(B_r(y)))$ is a nonnegative weak solution of (4.2.4) with data $\mu \in \mathfrak{M}^+(Q_r(y,s))$ and $u(s-r^2) \geq 0$, then there exists a constant C depending on N, Λ_1, Λ_2 such that*

$$u(y,s) \geq C \sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N}, \tag{4.5.10}$$

where $r_k = 4^{-k} r$.

Proof. It is enough to show that for $\rho \in (0,r)$

$$\frac{\mu(Q_{\rho/8}(y, s - \frac{35}{128}\rho^2))}{\rho^N} \leq c_1 (\inf_{Q_{\rho/4}(y,s)} u - \inf_{Q_\rho(y,s)} u). \tag{4.5.11}$$

By [50, Theorem 6.18, p. 122], we have for any $\theta \in (0, 1 + 2/N)$,

$$\left(\fint_{Q_{\rho/4}(y, s-\rho^2/4)} (u-a)^\theta \right)^{1/\theta} \leq c_2 (b-a), \tag{4.5.12}$$

where $b = \inf_{Q_{\rho/4}(y,s)} u$, $a = \inf_{Q_\rho(y,s)} u$ and a constant c_2 depends on $N, \Lambda_1, \Lambda_2, \theta$.

Let $\eta \in C_c^\infty(Q_\rho(y,s))$ such that $0 \leq \eta \leq 1$, supp$\eta \subset Q_{\rho/4}(y, s - \frac{1}{4}\rho^2)$, $\eta = 1$ in $Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)$ and $|\nabla \eta| \leq c_3/\rho^2$, $|\eta_t| \leq c_3/\rho^2$ where $c_3 = c_3(N)$. We have

$$\begin{aligned}
\mu(Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)) &\leq \int_{Q_\rho(y,s)} \eta^2 d\mu(x,t) \\
&= \int_{Q_\rho(y,s)} u_t \eta^2 dxdt + 2 \int_{Q_\rho(y,s)} \eta A(x,t,\nabla u) \nabla \eta dxdt \\
&= -2 \int_{Q_\rho(y,s)} (u-a) \eta_t \eta dxdt + 2 \int_{Q_\rho(y,s)} \eta A(x,t,\nabla u) \nabla \eta dxdt \\
&\leq c_3 r^{-2} \int_{Q_{\rho/4}(y, s - \frac{1}{4}\rho^2)} (u-a) dxdt + 2\Lambda_1 \int_{Q_\rho(y,s)} \eta |\nabla u| |\nabla \eta| dxdt \\
&\leq c_4 r^N (b-a) + c_4 \int_{Q_\rho(y,s)} \eta |\nabla u| |\nabla \eta| dxdt.
\end{aligned}$$

Here we used (4.5.12) with $\theta = 1$ in the last inequality. It remains to show that

$$\int_{Q_r(y,s)} \eta |\nabla u| |\nabla \eta| dxdt \leq c_5 r^N (b-a). \tag{4.5.13}$$

First, we verify that for $\varepsilon \in (0,1)$

$$\int_{Q_\rho(y,s)} |\nabla u|^2 (u-a)^{-\varepsilon - 1} \eta^2 dxdt \leq c_6 \int_{Q_\rho(y,s)} (u-a)^{1-\varepsilon} \left(\eta |\eta_t| + |\nabla \eta|^2 \right) dxdt. \tag{4.5.14}$$

Indeed, for $\delta \in (0,1)$ we choose $\varphi = (u-a+\delta)^{-\varepsilon}\eta^2$ as test function in (4.2.4),

$$\begin{aligned}
0 &\leq \int_{Q_\rho(y,s)} u_t(u-a+\delta)^{-\varepsilon}\eta^2 dxdt + \int_{Q_\rho(y,s)} A(x,t,\nabla u)\nabla\left((u-a+\delta)^{-\varepsilon}\eta^2\right)dxdt \\
&\leq 2(1-\varepsilon)\int_{Q_\rho(y,s)}(u-a+\delta)^{1-\varepsilon}|\eta_t|\eta dxdt - \varepsilon\Lambda_2\int_{Q_\rho(y,s)}|\nabla u|^2(u-a+\delta)^{-\varepsilon-1}\eta^2 dxdt \\
&\quad + 2\Lambda_1\int_{Q_\rho(y,s)}\eta|\nabla u|(u-a+\delta)^{-\varepsilon}|\nabla\eta|dxdt.
\end{aligned}$$

So, we deduce (4.5.14) from using the Holder inequality and letting $\delta \to 0$.
Therefore, for $\varepsilon \in (0, 2/N)$ using the Holder inequality and we get

$$\begin{aligned}
&\int_{Q_r(y,s)}\eta|\nabla u||\nabla\eta|dxdt \\
&\leq \left(\int_{Q_\rho(y,s)}|\nabla u|^2(u-a)^{-\varepsilon-1}\eta^2 dxdt\right)^{1/2}\left(\int_{Q_\rho(y,s)}(u-a)^{\varepsilon+1}|\nabla\eta|^2 dxdt\right)^{1/2} \\
&\leq c_7\left(\int_{Q_\rho(y,s)}(u-a)^{1-\varepsilon}\left(\eta|\eta_t|+|\nabla\eta|^2\right)dxdt\right)^{1/2}\left(\int_{Q_\rho(y,s)}(u-a)^{\varepsilon+1}|\nabla\eta|^2 dxdt\right)^{1/2} \\
&\leq c_8\rho^{-2}\left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)}(u-a)^{1-\varepsilon}dxdt\right)^{1/2}\left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)}(u-a)^{\varepsilon+1}dxdt\right)^{1/2}.
\end{aligned}$$

Consequently, we get (4.5.11) from (4.5.12). ■

Proof of Theorem 4.2.3. Let $\mu_n \in (C_c^\infty(\Omega_T))^+, \sigma_n \in (C_c^\infty(\Omega))^+$ be in the proof of Theorem 4.2.1. Let u_n be a weak solution of equation

$$\begin{cases}
(u_n)_t - div(A(x,t,\nabla u_n)) = \mu_n \text{ in } \Omega_T, \\
u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\
u_n(0) = \sigma_n \quad \text{on } \Omega.
\end{cases}$$

As the proof of Theorem 4.2.1, thanks to Theorem 4.5.4 we get By Remark for any $Q_r(y,s) \subset \Omega \times (-\text{diam}(\Omega), T)$ and $r_k = 4^{-k}r$

$$u_n(y,s) \geq c_1\sum_{k=0}^{\infty}\frac{\mu_n(Q_{r_k/8}(y,s-\frac{35}{128}r_k^2))}{r_k^N} + c_1\sum_{k=0}^{\infty}\frac{(\sigma_n\otimes\delta_{\{t=0\}})(Q_{r_k/8}(y,s-\frac{35}{128}r_k^2))}{r_k^N}.$$

Finally, by Proposition 4.3.5 and Theorem 4.3.6 we get the results. ■

Remark 4.5.5 *If $u \in L^q(\Omega_T)$ satisfies (4.2.8) then $\mathcal{G}_2[\chi_E\mu] \in L^q(\mathbb{R}^{N+1})$ and $\mathbf{G}_{\frac{2}{q}}[\chi_F\sigma] \in L^q(\mathbb{R}^N)$ for every $E \subset\subset \Omega \times [0,T)$ and $F \subset\subset \Omega$. Indeed, for $E \subset\subset \Omega \times [0,T)$, $\varepsilon = dist(E, (\Omega \times (0,T)) \cup (\Omega \times \{t = T\})) > 0$, we can see that for any $(y,s) \in \Omega_T$, $r_k = 4^{-k}\varepsilon/4$*

$$u(y,s) \geq c_1\sum_{k=0}^{\infty}\frac{\tilde{\mu}(E\cap Q_{r_k/8}(y,s-\frac{35}{128}r_k^2))}{r_k^N}, \tag{4.5.15}$$

where $\tilde{\mu} = \mu + \sigma \otimes \delta_{\{t=0\}}$.
Moreover, for any $(y, s) \notin \Omega_T$

$$\sum_{k=0}^{\infty} \frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} = 0.$$

Thus,

$$\begin{aligned}
\infty &> \int_{\mathbb{R}^{N+1}} \sum_{k=0}^{\infty} \left(\frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \right)^q dyds \\
&= \int_{\mathbb{R}^N} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left(\frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \right)^q dsdy \\
&\geq \int_{\mathbb{R}^N} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_{r_k/8}(y, s))}{r_k^N} \right)^q dsdy \\
&\geq c_2 \int_{\mathbb{R}^{N+1}} \int_0^{\varepsilon/64} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_\rho(y, s))}{\rho^N} \right)^q \frac{d\rho}{\rho} dsdy \\
&\geq c_3(\varepsilon) \int_{\mathbb{R}^{N+1}} (\mathcal{G}_2[\tilde{\mu}\chi_E])^q \, dsdy.
\end{aligned}$$

Thus, from Proposition 4.4.19, we get the results.

Proof of Theorem 4.2.5. Set $D_n = B_n(0) \times (-n^2, n^2)$. For $n \geq 4$, by Theorem 4.2.1, there exists a renormalized solution u_n to problem

$$\begin{cases} (u_n)_t - div(A(x, t, \nabla u_n)) = \chi_{D_{n-1}} \omega \ \text{ in } D_n, \\ u_n = 0 \ \text{ on } \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \ \text{ on } B_n(0). \end{cases}$$

relative to a decomposition (f_n, g_n, h_n) of $\chi_{D_{n-1}} \omega_0$ satisfying

$$-K \mathbb{I}_2[\omega^-](x, t) \leq u_n(x, t) \leq K \mathbb{I}_2[\omega^+](x, t) \ \ \forall \ (x, t) \in D_n. \tag{4.5.16}$$

From the proof of Theorem 4.2.1 and Remark 4.3.9, we can assume that u_n satisfies (4.3.14) and (4.3.15) in Proposition 4.3.16 with $1 < q_0 < \frac{N+2}{N}$, $L \equiv 0$ and

$$||f_n||_{L^1(D_i)} + ||g_n||_{L^2(D_i)} + |||h_n| + |\nabla h_n|||_{L^2(D_i)} \leq 2|\omega|(D_{i+1}) \tag{4.5.17}$$

for any $i = 1, ..., n-1$ and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.
On the other hand, by Lemma 4.4.26 we have for any $s \in (1, \frac{N+2}{N})$

$$\begin{aligned}
\int_{D_m} |u_n|^s dxdt &\leq K^s \int_{D_m} (I_2[|\omega|])^s dxdt \\
&\leq K^s \int_{\tilde{Q}_{4m}(x_0, t_0)} (I_2[|\omega|])^s dxdt \\
&\leq c_1 M m^{N+2},
\end{aligned} \tag{4.5.18}$$

for $n \geq m \geq |x_0| + |t_0|^{1/2}$. Consequently, we can apply Proposition 4.3.17 and obtain that u_n converges to some u in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$.
Since for any $\alpha \in (0, 1/2)$

$$\int_{D_m} \frac{|\nabla u_n|^2}{(|u_n| + 1)^{\alpha+1}} dxdt \leq C_m(\alpha) \quad \forall\, n \geq m,$$

thus using (4.5.18) and Holder inequality, we get for any $1 \leq s_1 < \frac{N+2}{N+1}$

$$\int_{D_m} |\nabla u_n|^{s_1} dxdt \leq C_m(s_1) \quad \text{for all} \quad n \geq m \geq |x_0| + |t_0|^{1/2}.$$

This yields $u_n \to u$ in $L^{s_1}_{\text{loc}}(\mathbb{R}; W^{1,s_1}_{\text{loc}}(\mathbb{R}^N))$.
Take $\varphi \in C^\infty_c(\mathbb{R}^{N+1})$ and $m_0 \in \mathbb{N}$ with $\text{supp}(\varphi) \subset D_{m_0}$, we have for $n \geq m_0 + 1$

$$-\int_{\mathbb{R}^{N+1}} u_n \varphi_t dxdt + \int_{\mathbb{R}^{N+1}} A(x, t, \nabla u_n) \nabla \varphi dxdt = \int_{\mathbb{R}^{N+1}} \varphi d\omega$$

Letting $n \to \infty$, we conclude that u is a distribution solution to problem (4.2.6) with data $\mu = \omega$ which satisfies (4.2.9).
Claim 1. If $\omega \geq 0$. By Theorem 4.2.3, we have for $n \geq 4^{k_0+1}$, $(y, s) \in B_{4^{k_0}} \times (0, n^2)$

$$u_n(y, s) \geq c_2 \sum_{k=0}^{\infty} \frac{\omega(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2) \cap D_{n-1})}{r_k^N},$$

where $r_k = 4^{-k+k_0}$. This gives

$$u_n(y, s) \geq c_2 \sum_{k=-k_0}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(y, s - 35 \times 2^{-4k-7}) \cap B_{n-1}(0) \times (0, (n-1)^2))}{2^{-2Nk}}.$$

Letting $n \to \infty$ and $k_0 \to \infty$ we have (4.2.10). Finally, thanks to Proposition 4.4.8 and Theorem 4.4.2, we will assert (4.2.11) if we show that for $q > \frac{N+2}{N}$

$$\int_{\mathbb{R}} \left(\sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dxdt \geq c_3 \int_{\mathbb{R}} \int_0^{+\infty} \left(\frac{\omega(\tilde{Q}_\rho(x, t))}{\rho^N} \right)^q \frac{d\rho}{\rho} dxdt.$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dxdt \\
&\geq \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left(\frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}} \right)^q dtdx \\
&= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left(\frac{\omega(\tilde{Q}_{2^{-2k-3}}(x, t))}{2^{-2Nk}} \right)^q dt \\
&\geq c_4 \int_{\mathbb{R}^{N+1}} \int_0^{+\infty} \left(\frac{\omega(\tilde{Q}_\rho(x, t))}{\rho^N} \right)^q \frac{d\rho}{\rho} dxdt.
\end{aligned}$$

Claim 2. If A is independent of space variable x and (4.2.27) is satisfied. By Remark 4.5.3 we get for any $(x,t) \in D_{n/4}$

$$|\nabla u_n(x,t)| \leq c_5 \mathbb{I}_1[|\omega|](x,t).$$

Letting $n \to \infty$, we get (4.2.12).
Claim 3. If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark (4.5.2) we can assume that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (4.2.5). The proof is complete. ■

Remark 4.5.6 *If $\omega \in \mathfrak{M}_b(\mathbb{R}^{N+1})$ then u satisfies*

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\mathbb{R}^{N+1})} \leq C(N,\Lambda_1,\Lambda_2)|\omega|(\mathbb{R}^{N+1}).$$

Moreover, $I_2[|\omega|] \in L^{\frac{N+2}{N},\infty}(\mathbb{R}^{N+1})$ and $I_2[|\omega|] < \infty$ a.e in $\mathbb{R}^{N+1}$.

4.6 Quasilinear Lane-Emden Type Parabolic Equations

4.6.1 Quasilinear Lane-Emden Parabolic Equations in Ω_T

To prove Theorem 4.2.8 we need the following proposition which was proved in [6].

Proposition 4.6.1 *Assume O is an open subset of $\mathbb{R}^{N+1}$. Let $p > 1$ and $\mu \in \mathfrak{M}^+(O)$. If μ is absolutely continuous with respect to $Cap_{2,1,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(W_p^{2,1}(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Moreover, if $\mu \in \mathfrak{M}_b^+(O)$ then $||\mu_n - \mu||_{\mathfrak{M}_b(O)} \to 0$ as $n \to \infty$.*

Remark 4.6.2 *By Theorem 4.4.17, $W_p^{2,1}(\mathbb{R}^{N+1}) = \mathcal{L}_2^p(\mathbb{R}^{N+1})$, it follows $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*$. Note that $||\mu_n||_{\left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*} = ||\check{\mathcal{G}}_2[\mu_n]||_{L^{p'}(\mathbb{R}^{N+1})}$. So $\check{\mathcal{G}}_2[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$. Consequently, from (4.4.17) in Proposition 4.4.8, we obtain $\mathbb{I}_2^R[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$ and $R > 0$. In particular, $\mathbb{I}_2[\mu_n] \in L^{p'}_{loc}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$.*

Remark 4.6.3 *As in the proof of Theorem 2.5 in [16], we can prove a general version of Proposition 4.6.1, that is : for $p > 1$, if μ is absolutely continuous with respect to $Cap_{\mathcal{G}_\alpha,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Furthermore, $\mathbb{I}_\alpha[\mu_n] \in L^{p'}_{loc}(\mathbb{R}^{N+1})$ for all $n \in \mathbb{N}$. Besides, we also obtain that for $\mu \in \mathfrak{M}_b(O)$ is absolutely continuous with respect to $Cap_{\mathcal{G}_\alpha,p}$ in O if and only if $\mu = f + \nu$ where $f \in L^1(O)$ and $\nu \in \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$.*

Proof of Theorem 4.2.8. First, assume that $\sigma \in L^1(\Omega)$. Because μ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$, so are μ^+ and μ^-. Applying Proposition 4.6.1 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathfrak{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2[\mu_{1,n}], \mathbb{I}_2[\mu_{2,n}] \in L^q(\Omega_T)$.

For $i = 1,2$, set $\tilde{\mu}_{i,1} = \mu_{i,1}$ and $\tilde{\mu}_{i,j} = \mu_{i,j} - \mu_{i,j-1} \geq 0$, so $\mu_{i,n} = \sum_{j=1}^{n} \tilde{\mu}_{i,j}$. We write $\mu_{i,n} = \mu_{i,n,0} + \mu_{i,n,s}$, $\tilde{\mu}_{i,j} = \tilde{\mu}_{i,j,0} + \tilde{\mu}_{i,j,s}$ with $\mu_{i,n,0}, \tilde{\mu}_{i,n,0} \in \mathfrak{M}_0(\Omega_T)$, $\mu_{i,n,s}, \tilde{\mu}_{i,n,s} \in \mathfrak{M}_s(\Omega_T)$. As in the proof of Theorem 4.2.1, for any $j \in \mathbb{N}$ and $i = 1,2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m,i,j,0} = (f_{m,i,j}, g_{m,i,j}, h_{m,i,j})$ and $\tilde{\mu}_{m,i,j,s}$ such that $f_{m,i,j}, g_{m,i,j}, h_{m,i,j} \in C_c^\infty(\Omega_T)$ and strongly converge to some $f_{i,j}, g_{i,j}, h_{i,j}$ in $L^1(\Omega_T)$, $L^2(\Omega_T, \mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively and $\tilde{\mu}_{m,i,j}, \tilde{\mu}_{m,i,j,s} \in C_c^\infty(\Omega_T)$ converge to $\tilde{\mu}_{i,j}, \tilde{\mu}_{i,j,s}$ resp. in the narrow topology with $\tilde{\mu}_{m,i,j} = \tilde{\mu}_{m,i,j,0} + \tilde{\mu}_{m,i,j,s}$ which satisfy $\tilde{\mu}_{i,j,0} = (f_{i,j}, g_{i,j}, h_{i,j})$ and $0 \leq \tilde{\mu}_{m,i,j} \leq \varphi_m * \tilde{\mu}_{i,j}$ and

$$||f_{m,i,j}||_{L^1(\Omega_T)} + \|g_{m,i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)} + ||h_{m,i,j}||_{L^2(0,T,H_0^1(\Omega))} + \mu_{m,i,j,s}(\Omega_T) \leq 2\tilde{\mu}_{i,j}(\Omega_T). \quad (4.6.1)$$

Here $\{\varphi_m\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$.
For any $n,k,m \in \mathbb{N}$, let $u_{n,k,m}, u_{1,n,k,m}, u_{2,n,k,m} \in W$ with $W = \{z : z \in L^2(0,T,H_0^1(\Omega)), z_t \in L^2(0,T,H^{-1}(\Omega))\}$ be solutions of problems

$$\begin{cases} (u_{n,k,m})_t - div(A(x,t,\nabla u_{n,k,m})) + T_k(|u_{n,k,m}|^{q-1}u_{n,k,m}) = \sum_{j=1}^{n}(\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}) \text{ in } \Omega_T, \\ u_{n,k,m} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_{n,k,m}(0) = T_n(\sigma^+) - T_n(\sigma^-) \quad \text{on } \Omega, \end{cases} \quad (4.6.2)$$

$$\begin{cases} (u_{1,n,k,m})_t - div(A(x,t,\nabla u_{1,n,k,m})) + T_k(u_{1,n,k,m}^q) = \sum_{j=1}^{n} \tilde{\mu}_{m,1,j} \text{ in } \Omega_T, \\ u_{1,n,k,m} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_{1,n,k,m}(0) = T_n(\sigma^+) \quad \text{in } \Omega, \end{cases} \quad (4.6.3)$$

$$\begin{cases} (u_{2,n,k,m})_t - div(\tilde{A}(x,t,\nabla u_{2,n,k,m})) + T_k(u_{2,n,k,m}^q) = \sum_{j=1}^{n} \tilde{\mu}_{m,2,j} \text{ in } \Omega_T, \\ u_{2,n,k,m} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_{2,n,k,m}(0) = T_n(\sigma^-) \quad \text{in } \Omega, \end{cases} \quad (4.6.4)$$

where $\tilde{A}(x,t,\xi) = -A(x,t,-\xi)$.
By Comparison Principle Theorem and Theorem 4.2.1, there holds, for any m,k the sequences $\{u_{1,n,k,m}\}_n$ and $\{u_{2,n,k,m}\}_n$ are increasing and

$$\begin{aligned} -K\mathbb{I}_2[T_n(\sigma^-) \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n} * \varphi_m] &\leq -u_{2,n,k,m} \leq u_{n,k,m} \leq u_{1,,n,k,m} \\ &\leq K\mathbb{I}_2[\mu_{1,n} * \varphi_m] + K\mathbb{I}_2[T_n(\sigma^+) \otimes \delta_{\{t=0\}}], \end{aligned}$$

where a constant K is in Theorem 4.2.1. Thus,

$$\begin{aligned} -K\mathbb{I}_2[T_n(\sigma^-) \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}] * \varphi_m &\leq -u_{2,n,k,m} \leq u_{n,k,m} \leq u_{1,,n,k,m} \\ &\leq K\mathbb{I}_2[\mu_{1,n}] * \varphi_m + K\mathbb{I}_2[T_n(\sigma^+) \otimes \delta_{\{t=0\}}]. \end{aligned}$$

Moreover,

$$\int_{\Omega_T} T_k(u_{i,n,k,m}^q) dxdt \leq \int_{\Omega_T} \varphi_m * \mu_{i,n} dxdt + |\sigma|(\Omega) \leq |\mu|(\Omega_T) + |\sigma|(\Omega).$$

As in [14, Proof of Lemma 5.3], thanks to Proposition 4.3.5 and Theorem 4.3.6, there exist subsequences of $\{u_{n,k,m}\}_m$ $\{u_{1,n,k,m}\}_m$, $\{u_{2,n,k,m}\}_m$, still denoted them, converging to renormalized solutions $u_{n,k}$ $u_{1,n,k}$, $u_{2,n,k}$ of equations (4.6.2) with data $\mu_{1,n}-\mu_{2,n}$, $u_{n,k}(0)=T_n(\sigma^+)-T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{1,j}-\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{1,j}-\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{1,j}-\sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$, (4.6.3) with data $\mu_{1,n}$, $u_{1,n,k}(0)=T_n(\sigma^+)$ and the decomposition $(\sum_{j=1}^n f_{1,j},\ \sum_{j=1}^n g_{1,j},\ \sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$, (4.6.4) with data $\mu_{2,n}$, $u_{2,n,k}(0)=T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{2,j},\ \sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively, which satisfy

$$\begin{aligned}-K\mathbb{I}_2[T_n(\sigma^-)\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu_{2,n}]&\leq -u_{2,n,k}\leq u_{n,k}\leq u_{1,n,k}\\ &\leq K\mathbb{I}_2[\mu_{1,n}]+K\mathbb{I}_2[T_n(\sigma^+)\otimes\delta_{\{t=0\}}].\end{aligned}$$

Next, as in [14, Proof of Lemma 5.4] since $I_2[\mu_{i,n}]\in L^q(\Omega_T)$ for any n, thanks to Proposition 4.3.5 and Theorem 4.3.6, there exist subsequences of $\{u_{n,k}\}_k$ $\{u_{1,n,k}\}_k$, $\{u_{2,n,k}\}_k$, still denoted them, converging to renormalized solutions u_n $u_{1,n}$, $u_{2,n}$ of equations

$$\begin{cases}(u_n)_t-div(A(x,t,\nabla u_n))+|u_n|^{q-1}u_n=\mu_{1,n}-\mu_{2,n} & \text{in } \Omega_T,\\ u_n=0 & \text{on } \partial\Omega\times(0,T),\\ u_n(0)=T_n(\sigma^+)-T_n(\sigma^-) & \text{in } \Omega,\end{cases}\tag{4.6.5}$$

$$\begin{cases}(u_{1,n})_t-div(A(x,t,\nabla u_{1,n}))+u_{1,n}^q=\mu_{1,n} & \text{in } \Omega_T,\\ u_{1,n}=0 & \text{on } \partial\Omega\times(0,T),\\ u_{1,n}(0)=T_n(\sigma^+) & \text{in } \Omega,\end{cases}\tag{4.6.6}$$

$$\begin{cases}(u_{2,n})_t-div(\tilde{A}(x,t,\nabla u_{2,n}))+u_{2,n}^q=\mu_{2,n} & \text{in } \Omega_T,\\ u_{2,n}=0 & \text{on } \partial\Omega\times(0,T),\\ u_{2,n}(0)=T_n(\sigma^-) & \text{in } \Omega,\end{cases}\tag{4.6.7}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j}-\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{1,j}-\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{1,j}-\sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$, $(\sum_{j=1}^n f_{1,j},\ \sum_{j=1}^n g_{1,j},\ \sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$ and $(\sum_{j=1}^n f_{2,j},\ \sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively, which satisfy

$$\begin{aligned}-K\mathbb{I}_2[T_n(u_0^-)\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu_{2,n}]&\leq -u_{2,n}\leq u_n\leq u_{1,n}\\ &\leq K\mathbb{I}_2[\mu_{1,n}]+K\mathbb{I}_2[T_n(u_0^+)\otimes\delta_{\{t=0\}}].\end{aligned}$$

and the sequences $\{u_{1,n}\}_n$ and $\{u_{2,n}\}_n$ are increasing and

$$\int_{\Omega_T}u_{i,n}^q dxdt\leq|\mu|(\Omega_T)+|\sigma|(\Omega).$$

Note that from (4.6.1) we have

$$||f_{i,j}||_{L^1(\Omega_T)}+\|g_{i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)}+||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))}\leq 2\tilde{\mu}_{i,j}(\Omega_T)$$

which implies

$$\sum_{j=1}^n||f_{i,j}||_{L^1(\Omega_T)}+\sum_{j=1}^n||g_{i,j}||_{L^2(\Omega_T,\mathbb{R}^N)}+\sum_{j=1}^n||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))}\leq 2\mu_{i,n}(\Omega_T)\leq 2|\mu|(\Omega_T).$$

Finally, as in [14, Proof of Theorem 5.2] thanks to Proposition 4.3.5, Theorem 4.3.6 and Monotone Convergence Theorem there exist subsequences of $\{u_n\}_n$, $\{u_{1,n}\}_n$, $\{u_{2,n}\}_n$, still denoted them, converging to renormalized solutions u, u_1, u_2 of equations (4.6.5) with data μ, $u(0) = \sigma$ and the decomposition $(\sum_{j=1}^{\infty} f_{1,j} - \sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{1,j} - \sum_{j=1}^{\infty} g_{2,j}, \sum_{j=1}^{\infty} h_{1,j} - \sum_{j=1}^{\infty} h_{2,j})$ of μ_0, (4.6.6) with data μ^+, $u_1(0) = \sigma^+$ and the decomposition $(\sum_{j=1}^{\infty} f_{1,j}, \sum_{j=1}^{\infty} g_{1,j}, \sum_{j=1}^{\infty} h_{1,j})$ of μ_0^+, (4.6.7) with data μ^-, $u_2(0) = \sigma^-$ and the decomposition $(\sum_{j=1}^{\infty} f_{2,j}, \sum_{j=1}^{\infty} g_{2,j}, \sum_{j=1}^{\infty} h_{2,j})$ of μ_0^-, respectively and

$$-K\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu^-] \le -u_2 \le u \le u_1 \le K\mathbb{I}_2[\mu^+] + K\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}].$$

We now have remark : if $\sigma \equiv 0$ and $\mathrm{supp}(\mu) \subset \overline{\Omega} \times [a, T]$, $a > 0$, then $u = u_1 = u_2 = 0$ in $\Omega \times (0, a)$ since $u_{n,k} = u_{1,n,k} = u_{2,n,k} = 0$ in $\Omega \times (0, a)$.
Next, we will consider $\sigma \in \mathfrak{M}_b(\Omega)$ such that σ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}}, q'}$ in Ω. So, $\chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,q'}$ in $\Omega \times (-T, T)$. As above, we verify that there exists a renormalized solution u of

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + |u|^{q-1}u = \chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}} & \text{in } \Omega \times (-T, T), \\ u = 0 & \text{on } \partial\Omega \times (-T, T), \\ u(-T) = 0 & \text{on } \Omega, \end{cases} \tag{4.6.8}$$

satisfying $u = 0$ in $\Omega \times (-T, 0)$ and

$$-K\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu^-] \le u \le K\mathbb{I}_2[\mu^+] + K\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}].$$

Finally, from remark 4.3.11 we get the result. This completes the proof of the theorem. ∎

Proof of Theorem 4.2.9. Let $\{\mu_{n,i}\} \subset C_c^{\infty}(\Omega_T), \sigma_{i,n} \in C_c^{\infty}(\Omega)$ for $i = 1, 2$ be as in the proof of Theorem 4.2.1. We have $0 \le \mu_{n,1} \le \varphi_n * \mu^+, 0 \le \mu_{n,2} \le \varphi_n * \mu^-, 0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-$ for any $n \in \mathbb{N}$ where $\{\varphi_n\}$ and $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively.
We prove that the problem (4.2.2) has a solution with data $\mu = \mu_{n_0} = \mu_{n_0,1} - \mu_{n_0,2}, \sigma = \sigma_{n_0} = \sigma_{1,n_0} - \sigma_{2,n_0}$ for $n_0 \in \mathbb{N}$. Put

$$J = \Big\{ u \in L^q(\Omega_T) : u^+ \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}] \\ \text{and} \quad u^- \le \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}] \Big\}.$$

where $\max\{-\frac{N+2}{q'} + 2, 0\} < \delta < 2$.
Clearly, J is closed under the strong topology of $L^q(\Omega_T)$ and convex.
We consider a map $S : J \to J$ defined for each $v \in J$ by $S(v) = u$, where $u \in L^1(\Omega_T)$ is the unique renormalized solution of

$$\begin{cases} u_t - div(A(x,t,\nabla u)) = |v|^{q-1}v + \mu_{n_0,1} - \mu_{n_0,2} \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = \sigma_{1,n_0} - \sigma_{2,n_0} \quad \text{in } \Omega. \end{cases} \tag{4.6.9}$$

By Theorem 4.2.1, we have

$$\begin{aligned} u^+ &\leq K\mathbb{I}_2^{2T_0}[(v^+)^q] + K\mathbb{I}_2^{2T_0}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}], \\ u^- &\leq K\mathbb{I}_2^{2T_0}[(v^-)^q] + K\mathbb{I}_2^{2T_0}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}], \end{aligned}$$

where K is the constant in Theorem 4.2.1. Thus,

$$u^+ \leq K\left(\frac{qK}{q-1}\right)^q \mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] + K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}],$$

$$u^- \leq K\left(\frac{qK}{q-1}\right)^q \mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] + K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}].$$

Thus, thanks to Theorem 4.4.36 there exists $c_1 = c_1(N, K, \delta, q)$ such that if for every compact sets $E \subset \mathbb{R}^{N+1}$,

$$|\mu_{n_0,i}|(E) + (|\sigma_{i,n_0}| \otimes \delta_{\{t=0\}})(E) \leq c_1 \text{Cap}_{E_2^{2T_0,\delta},q'}(E). \tag{4.6.10}$$

then $\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\mathbb{R}^{N+1})$ and

$$\mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] \leq \frac{(q-1)^{q-1}}{(Kq)^q}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \ i = 1, 2.$$

which implies $u \in L^q(\Omega_T)$ and

$$\begin{aligned} u^+ &\leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,1} + \sigma_{1,n_n} \otimes \delta_{\{t=0\}}] \quad \text{and} \\ u^- &\leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}]. \end{aligned}$$

Now we assume that (4.6.10) is satisfied, so S is well defined. Therefore, if we can show that the map $S : J \to J$ is continuous and $S(J)$ is pre-compact under the strong topology of $L^q(\Omega_T)$ then by Schauder Fixed Point Theorem, S has a fixed point on J. Hence the problem (4.2.2) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
Now we show that **S is continuous**. Let $\{v_n\}$ be a sequence in J such that v_n converges strongly in $L^q(\Omega_T)$ to a function $v \in J$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^q(\Omega_T)$.

By Proposition 4.3.5, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Since

$$|u_n| \leq \sum_{i=1,2} \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\Omega_T) \quad \forall \ n \in \mathbb{N}$$

Applying Dominated Convergence Theorem, we have $u_n \to u$ in $L^q(\Omega_T)$. Hence, thanks to Theorem 4.3.6 we get $u = S(v)$.
Next we show that **S is pre-compact**. Indeed if $\{u_n\} = \{S(v_n)\}$ is a sequence in $S(J)$. By Proposition 4.3.5, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Again, using get Dominated Convergence Theorem we get $u_n \to u$ in $L^q(\Omega_T)$.

So **S is pre-compact**.
Next, thanks to Corollary 4.4.39 and Remark 4.4.40 we have

$$[\mu_{n,i}+\sigma_{i,n}\otimes\delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\leq c_2[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\quad\forall\, n\in\mathbb{N}, i=1,2,$$

for some $c_2=c_2(N,q)$.
In addition, by the proof of Corollary 4.4.39 we get

$$(c_3(T_0))^{-1}\mathrm{Cap}_{\mathcal{G}_2,q'}(E)\leq \mathrm{Cap}_{E_2^{2T_0,\delta},q'}(E)\leq c_3(T_0)\mathrm{Cap}_{\mathcal{G}_2,q'}(E)$$

for every compact set E with $\mathrm{diam}(E)\leq 2T_0$. Thus, there is $c_4=c_4(N,K,\delta,q,T_0)$ such that if

$$[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\leq c_4, \tag{4.6.11}$$

then (4.6.10) holds for any $n_0\in\mathbb{N}$.
Now we suppose that (4.6.11) holds, it is equivalent to (4.2.13) holding for some constant $C_1=C_1(T_0)$ by Remark 4.4.34. Therefore, for any $n\in\mathbb{N}$ there exists a renormalized solution u_n of

$$\begin{cases} (u_n)_t-div(A(x,t,\nabla u_n))=|u_n|^{q-1}u_n+\mu_{n,1}-\mu_{n,2} \text{ in } \Omega_T,\\ u_n=0 \qquad \text{on } \ \partial\Omega\times(0,T),\\ u_n(0)=\sigma_{1,n}-\sigma_{2,n} \quad \text{in } \ \Omega, \end{cases} \tag{4.6.12}$$

which satisfies

$$-\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,2}+\sigma_{2,n}\otimes\delta_{\{t=0\}}]\leq u_n\leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,1}+\sigma_{1,n}\otimes\delta_{\{t=0\}}].$$

Thus, for every $(x,t)\in\Omega_T$,

$$\begin{aligned} -\frac{qK}{q-1}\varphi_n*\mathbb{I}_2^{2T_0,\delta}[\mu^-](x,t)-\frac{qK}{q-1}\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^-\otimes\delta_{\{t=0\}}](.,t))(x)\leq u_n(x,t)\\ \leq\frac{qK}{q-1}\varphi_n*(\mathbb{I}_2^{2T_0,\delta}[\mu^-])(x,t)+\frac{qK}{q-1}\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^-\otimes\delta_{\{t=0\}}](.,t))(x). \end{aligned}$$

Since $\varphi_n*\mathbb{I}_2^{2T_0,\delta}[\mu^{\pm}](x,t),\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^{\pm}\otimes\delta_{\{t=0\}}](.,t))(x)$ converge to $\mathbb{I}_2^{2T_0,\delta}[\mu^{\pm}](x,t),\mathbb{I}_2^{2T_0,\delta}[\sigma^{\pm}\otimes\delta_{\{t=0\}}](x,t)$ in $L^q(\mathbb{R}^{N+1})$ as $n\to\infty$, respectively, so $|u_n|^q$ is equi-integrable.
By Proposition 4.3.5, there exists a subsequence of $\{u_n\}$, still denoted by its, converging to u a.e in Ω_T. It follows $|u_n|^{q-1}u_n\to|u|^{q-1}u$ in $L^1(\Omega_T)$.
Consequently, by Proposition 4.3.5 and Theorem 4.3.6, we obtain that u is a distribution (a renormalized solution if $\sigma\in L^1(\Omega)$) of (4.2.2) with data μ, σ, and satisfies (4.2.14). Furthermore, by Corollary 4.4.39 we have

$$\begin{aligned} &(c_5(T_0))^{-1}\left[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}\right]^q_{\mathfrak{M}^{\mathcal{G}_2,q'}}\\ &\qquad\leq\left[\left(\mathbb{I}_2^{2T_0,\delta}[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}]\right)^q\right]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\leq c_5(T_0)\left[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}\right]^q_{\mathfrak{M}^{\mathcal{G}_2,q'}} \end{aligned}$$

which implies $[|u|^q]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\leq c_4(T_0)$ and we get (4.2.15). This completes the proof of the Theorem. ∎

Remark 4.6.4 *In view of above proof, we can see that*

i. *The Theorem 4.2.9 also holds when we replace assumption* (4.2.13) *by*

$$|\mu|(E) \leq C\,Cap_{\mathcal{H}_{2,q'}}(E) \quad \text{and} \quad |\sigma|(F) \leq C\,Cap_{\mathbf{I}_{\frac{2}{q}},q'}(F).$$

for every compact sets $E \subset \mathbb{R}^{N+1}, F \subset \mathbb{R}^N$ where $C = C(N\Lambda_1, \Lambda_2, q)$ is some a constant.

ii. *If $\sigma \equiv 0$ and $supp(\mu) \subset \overline{\Omega} \times [a,T]$, $a > 0$, then we can show that a solution u in Theorem 4.2.9 satisfies $u = 0$ in $\Omega \times (0,a)$ since we can replace the set E by E' :*

$$E' = \Big\{ u \in L^q(\Omega_T) : u = 0 \text{ in } \Omega \times (0,a) \text{ and } u^+ \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1} + \sigma_{1,n_0} \otimes \delta_{\{t=0\}}]$$
$$\text{and} \quad u^- \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}] \Big\}.$$

4.6.2 Quasilinear Lane-Emden Parabolic Equations in $\mathbb{R}^N \times (0,\infty)$ and $\mathbb{R}^{N+1}$

This section is devoted to proofs of Theorem 4.2.12 and 4.2.14.

Proof of the Theorem 4.2.12. Since ω is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\mathbb{R}^{N+1}$, $|\omega|$ is too. Set $D_n = B_n(0) \times (-n^2, n^2)$. From the proof of Theorem 4.2.8, there exist renormalized solutions u_n, v_n of

$$\begin{cases} (u_n)_t - \text{div}(A(x,t,\nabla u_n)) + |u_n|^{q-1}u_n = \chi_{D_n}\omega & \text{in } D_n, \\ u_n = 0 \quad \text{on } \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{in } B_n(0), \end{cases}$$

and

$$\begin{cases} (v_n)_t - \text{div}(A(x,t,\nabla v_n)) + v_n^q = \chi_{D_n}|\omega| & \text{in } D_n, \\ v_n = 0 \quad \text{on } \partial B_n(0) \times (-n^2, n^2), \\ v_n(-n^2) = 0 \quad \text{in } B_n(0), \end{cases}$$

relative to decompositions (f_n, g_n, h_n) of $\chi_{D_n}\omega_0$ and $(\overline{f}_n, \overline{g}_n, \overline{h}_n)$ of $\chi_{B_n(0)\times(0,n^2)}|\omega_0|$, satisfied (4.3.14), (4.3.15) in Proposition 4.3.16 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and μ is replaced by $\chi_{D_n}\omega$ and $\chi_{D_n}|\omega|$ respectively. Moreover, there hold

$$-KI_2[\omega^-] \leq u_n \leq KI_2[\omega^+], \; 0 \leq v_n \leq KI_2[|\omega|] \text{ in } D_n, \tag{4.6.13}$$

and $v_{n+1} \geq v_n$, $|u_n| \leq v_n$ in D_n.

By Remark 4.3.9, we can assume that

$$||f_n||_{L^1(D_i)} + ||g_n||_{L^2(D_i,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(D_i)} \leq 2|\omega|(D_{i+1}) \text{ and}$$
$$||\overline{f}_n||_{L^1(D_i)} + ||\overline{g}_n||_{L^2(D_i,\mathbb{R}^N)} + |||\overline{h}_n| + |\nabla \overline{h}_n|||_{L^2(D_i)} \leq 2|\omega|(D_{i+1}),$$

for any $i = 1, ..., n-1$ and $h_n, \overline{h}_n$ are convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. On the other hand, since u_n, v_n satisfy (4.3.14) in Proposition 4.3.16 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and thanks to Holder inequality : for any $\varepsilon \in (0,1)$

$$(|u_n| + 1)^{q_0} \leq \varepsilon |u_n|^q + c_1(\varepsilon) \quad \text{and} \quad (|v_n| + 1)^{q_0} \leq \varepsilon |v_n|^q + c_1(\varepsilon).$$

Thus we get

$$\int_{D_i}|u_n|^q dxdt+\int_{D_i}|u_n|^{q_0}dxdt+\int_{D_i}v_n^q dxdt+\int_{D_i}v_n^{q_0}dxdt\le C(i)+c_2|\omega|(D_{i+1}). \tag{4.6.14}$$

for $i=1,...,n-1$, where the constant $C(i)$ depends on $N,\Lambda_1,\Lambda_2,q_0,q$ and i.
Consequently, we can apply Proposition 4.3.17 with $\mu_n=-|u_n|^{q-1}u_n+\chi_{D_n}\omega,-v_n^q+\chi_{D_n}|\omega|$ and obtain that there are subsequences of u_n,v_n, still denoted by them, converging to some u,v in $L^1_{loc}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$. So, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha>0$ and $u\in L^q_{loc}(\mathbb{R}^{N+1})$ satisfies (4.2.17). In addition, using Holder inequality we get $u\in L^\gamma_{\text{loc}}(\mathbb{R};W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1\le\gamma<\frac{2q}{q+1}$.
Thanks to (4.6.14) and Monotone Convergence Theorem we get $v_n\to v$ in $L^q_{loc}(\mathbb{R}^{N+1})$. After, we also have $u_n\to u$ in $L^q_{loc}(\mathbb{R}^{N+1})$ by $|u_n|\le v_n$ and Dominated Convergence Theorem. Consequently, u is a distribution solution of problem (4.2.16) which satisfies (4.2.17). If $\omega=\mu+\sigma\otimes\delta_{\{t=0\}}$ with $\mu\in\mathfrak{M}(\mathbb{R}^N\times(0,\infty))$ and $\sigma\in\mathfrak{M}(\mathbb{R}^N)$, then by the proof of Theorem 4.2.8 we can assume that $u_n=0$ in $B_n(0)\times(-n^2,0)$. So, $u=0$ in $\mathbb{R}^N\times(-\infty,0)$. Therefore, clearly $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to (4.2.18).
This completes the proof of the theorem. ∎

Proof of the Theorem 4.2.14. By the proof of Theorem 4.2.9 and Remark 4.6.4, 4.4.34, there exists a constant $c_1=c_1(N,q,\Lambda_1,\Lambda_2)$ such that if ω satisfy for every compact set $E\subset\mathbb{R}^{N+1}$,

$$|\omega|(E)\le c_1\text{Cap}_{\mathcal{H}_2,q'}(E), \tag{4.6.15}$$

then there is a renormalized solution u_n of

$$\begin{cases} (u_n)_t-\text{div}(A(x,t,\nabla u_n))=|u_n|^{q-1}u_n+\chi_{D_n}\omega \text{ in } D_n\\ u_n=0 \quad \text{ on } \partial B_n(0)\times(-n^2,n^2),\\ u_n(-n^2)=0 \quad\text{in}\quad B_n(0),\end{cases}$$

relative to a decomposition (f_n,g_n,h_n) of $\chi_{D_n}\omega_0$, satisfying (4.3.14), (4.3.15) in Proposition 4.3.16 with $q_0=q$, $L\equiv 0$ and μ is replaced by $|u_n|^{q-1}u_n+\chi_{D_n}\omega$ and

$$-\frac{qK}{q-1}\mathbb{I}_2[\omega^-](x,t)\le u_n\le\frac{qK}{q-1}\mathbb{I}_2[\omega^+](x,t) \tag{4.6.16}$$

for a.e (x,t) in D_n and $I_2[\omega^\pm]\in L^q_{loc}(\mathbb{R}^{N+1})$.
Besides, thanks to Remark 4.3.9, we can assume that f_n,g_n,h_n satisfies (4.5.17) in proof of Theorem (4.2.5) and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.
Consequently, we can apply Proposition 4.3.17 and obtain that there exist a subsequence of u_n, still denoted by it, converging to some u a.e in $\mathbb{R}^{N+1}$ and in $L^1_{loc}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$. Also, $u_n\to u$ in $L^q_{\text{loc}}(\mathbb{R}^{N+1})$ by Dominated Convergence Theorem, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha>0$. Using Holder inequality we get $u\in L^\gamma_{\text{loc}}(\mathbb{R};W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1\le\gamma<\frac{2q}{q+1}$.
Thus we obtain that u is a distribution solution of (4.2.20) which satisfies (4.2.21). Since (4.6.15) holds, thus by Theorem 4.4.36 we get

$$c_2^{-1}\left[|\omega|\right]^q_{\mathfrak{M}^{\mathcal{H}_2,q'}}\le\left[(\mathbb{I}_2[|\omega|])^q\right]_{\mathfrak{M}^{\mathcal{H}_2,q'}}\le c_2\left[|\omega|\right]^q_{\mathfrak{M}^{\mathcal{H}_2,q'}},$$

so we have $[|u|^q]_{\mathfrak{M}^{\mathcal{H}_{2,q'}}} \le c_3$. It follows (4.2.23).
If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark 4.6.4 we can assume that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (4.2.22).
This completes the proof of the theorem. ■

4.7 Interior Estimates and Boundary Estimates for Parabolic Equations

In this section we always assume that $u \in C(-T,T,L^2(\Omega)) \cap L^2(-T,T,H_0^1(\Omega))$ is a solution to equation (4.2.4) in $\Omega \times (-T,T)$ with $\mu \in L^2(\Omega \times (-T,T))$ and $u(-T) = 0$. We extend u by zero to $\Omega \times (-\infty, -T)$, clearly u is a solution to equation

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = \chi_{(-T,T)}(t)\mu & \text{in } \Omega \times (-\infty, T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (-\infty, T). \end{cases} \tag{4.7.1}$$

4.7.1 Interior Estimates

For each ball $B_{2R} = B_{2R}(x_0) \subset\subset \Omega$ and $t_0 \in (-T,T)$, one considers the unique solution

$$w \in C(t_0 - 4R^2, t_0; L^2(B_{2R})) \cap L^2(t_0 - 4R^2, t_0; H^1(B_{2R})) \tag{4.7.2}$$

to the following equation

$$\begin{cases} w_t - div\,(A(x,t,\nabla w)) = 0 \textit{ in } Q_{2R}, \\ w = u \qquad \textit{on} \;\; \partial_p Q_{2R}, \end{cases} \tag{4.7.3}$$

where $Q_{2R} = B_{2R} \times (t_0 - 4R^2, t_0)$ and $\partial_p Q_{2R} = \big(\partial B_{2R} \times (t_0 - 4R^2, t_0)\big) \cup \big(B_{2R} \times \{t = t_0 - 4R^2\}\big)$.

Theorem 4.7.1 *There exist constants $\theta_1 > 2$, $\beta_1 \in (0, \frac{1}{2}]$ and C_1, C_2, C_3 depending on N, Λ_1, Λ_2 such that the following estimates are true*

$$\fint_{Q_{2R}} |\nabla u - \nabla w| dxdt \le C_1 \frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{4.7.4}$$

$$\left(\fint_{Q_{\rho/2}(y,s)} |\nabla w|^{\theta_1} dxdt \right)^{\frac{1}{\theta_1}} \le C_2 \fint_{Q_\rho(y,s)} |\nabla w| dxdt, \tag{4.7.5}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w - \overline{w}_{Q_{\rho_1}(y,s)}|^2 dxdt \right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2}\right)^{\beta_1} \left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt \right)^{1/2}, \tag{4.7.6}$$

and

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2}\right)^{\beta_1 - 1} \left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \tag{4.7.7}$$

for any $Q_\rho(y,s) \subset Q_{2R}$, and $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{2R}$.

Proof. Inequalities (4.7.4), (4.7.5) and (4.7.6) were proved by Duzaar and Mingione in [27]. So, it remains to prove (4.7.7) in case $\rho_1 \leq \rho_2/2$. By interior Caccioppoli inequality we have

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt\right)^{1/2} \leq \frac{c_1}{\rho_1}\left(\fint_{Q_{2\rho_1}(y,s)} |w - \overline{w}_{Q_{2\rho_1}(y,s)}|^2 dxdt\right)^{1/2}.$$

On the other hand, by a Sobolev inequality there holds

$$\left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt\right)^{1/2} \leq c_2\rho_2\left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt\right)^{1/2}.$$

Therefore, (4.7.7) follows from (4.7.6). ∎

Corollary 4.7.2 *Let β_1 be the constant in Theorem 4.7.1. For $2-\beta_1 < \theta < N+2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, \theta) > 0$ such that for any $B_\rho(y) \subset B_{\rho_0}(y) \subset\subset \Omega$, $s \in (-T,T)$*

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \leq C\rho^{N+3-\theta}\left(\left(\frac{T_0}{\rho_0}\right)^{N+3-\theta} + 1\right) ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}. \tag{4.7.8}$$

Proof. Take $B_{\rho_2}(y) \subset\subset \Omega$ and $s \in (-T,T)$. For any $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s)$ with $\rho_1 \leq \rho_2/2$, we take w as in Theorem 4.7.1 with $Q_{2R} = Q_{\rho_2}(y,s)$. Thus,

$$\int_{Q_{\rho_1}(y,s)} |\nabla w| dxdt \leq c_1\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2}(y,s)} |\nabla w| dxdt,$$

$$\int_{Q_{\rho_2}(y,s)} |\nabla u - \nabla w| dxdt \leq c_2\rho_2|\mu|(Q_{\rho_2}(y,s)),$$

and we also have

$$c_3^{-1}\int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt \leq \int_{Q_{\rho_2}(y,s)} |\nabla w| dxdt \leq c_3\int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt.$$

It follows that

$$\begin{aligned}\int_{Q_{\rho_1}(y,s)} |\nabla u| dxdt &\leq \int_{Q_{\rho_1}(y,s)} |\nabla w| dxdt + \int_{Q_{\rho_1}(y,s)} |\nabla u - \nabla w| dxdt \\ &\leq c_4\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)} |\nabla w| dxdt + \int_{Q_{\rho_2}(y,s)} |\nabla u - \nabla w| dxdt \\ &\leq c_5\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt + c_5\rho_2|\mu|(Q_{\rho_2}(y,s)).\end{aligned}$$

This implies

$$\int_{Q_{\rho_1}(y,s)} |\nabla u| dxdt \leq c_5\left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1}\int_{Q_{\rho_2}(y,s)} |\nabla u| dxdt + c_5\rho_2^{N+3-\theta}||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Since $N+3-\beta < N+\beta_1+1$, applying [50, Lemma 4.6, page 54] we obtain

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \leq c_6 \left(\frac{\rho}{\rho_0}\right)^{N+3-\theta} ||\nabla u||_{L^1(\Omega\times(-T,T))} + c_6 \rho^{N+3-\theta} ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))},$$

for any $B_\rho(y) \subset B_{\rho_0}(y) \subset\subset \Omega$, $s \in (-T,T)$. On the other hand, by Remark 4.3.2

$$||\nabla u||_{L^1(\Omega\times(-T,T))} \leq c_7 T_0 |\mu|(\Omega \times (-T,T)) \leq c_8 T_0^{N+3-\theta} ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Hence, we get the desired result. ■

To continue, we consider the unique solution

$$v \in C(t_0-R^2, t_0; L^2(B_R)) \cap L^2(t_0-R^2, t_0; H^1(B_R)) \tag{4.7.9}$$

to the following equation

$$\begin{cases} v_t - div\left(\overline{A}_{B_R(x_0)}(t,\nabla v)\right) = 0 \text{ in } Q_R, \\ v = w \qquad on \;\; \partial_p Q_R, \end{cases} \tag{4.7.10}$$

where $Q_R = B_R(x_0) \times (t_0 - R^2, t_0)$ and $\partial_p Q_R = \left(\partial B_R \times (t_0-R^2,t_0)\right) \cup \left(B_R \times \left\{t = t_0 - R^2\right\}\right)$.

Lemma 4.7.3 *Let θ_1 be the constant in Theorem 4.7.1. There exist constants $C_1 = C_1(N,\Lambda_1,\Lambda_2)$ and $C_2 = C_2(\Lambda_1,\Lambda_2)$ such that*

$$\left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2} \leq C_1 [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt, \tag{4.7.11}$$

with $s_1 = \frac{2\theta_1}{\theta_1 - 2}$ and

$$C_2^{-1} \int_{Q_R} |\nabla v|^2 dxdt \leq \int_{Q_R} |\nabla w|^2 dxdt \leq C_2 \int_{Q_R} |\nabla v|^2 dxdt. \tag{4.7.12}$$

Proof. We can choose $\varphi = w - v$ as a test function for equations (4.7.3), (4.7.10) and since

$$\int_{Q_R} w_t (w-v) dxdt - \int_{Q_R} v_t (w-v) dxdt = \frac{1}{2} \int_{B_R} (w-v)^2(t_0) dx \geq 0,$$

we find

$$-\int_{Q_R} \overline{A}_{B_R(x_0)}(t,\nabla v) \nabla (w-v) dxdt \leq -\int_{Q_R} A(x,t,\nabla w) \nabla (w-v) dxdt.$$

By using inequalities (4.1.2) and (4.1.3) together with Holder's inequality we get

$$c_1^{-1} \int_{Q_R} |\nabla v|^2 dxdt \leq \int_{Q_R} |\nabla w|^2 dxdt \leq c_1 \int_{Q_R} |\nabla v|^2 dxdt, \tag{4.7.13}$$

and we also have

$$\begin{aligned}\Lambda_2 \int_{Q_R} |\nabla w - \nabla v|^2 dxdt &\le \int_{Q_R} \left(\overline{A}_{B_R(x_0)}(t,\nabla w) - \overline{A}_{B_R(x_0)}(t,\nabla v)\right)(\nabla w - \nabla v)\, dxdt \\ &\le \int_{Q_R} \left(\overline{A}_{B_R(x_0)}(t,\nabla w) - A(x,t,\nabla w)\right)(\nabla w - \nabla v)\, dxdt \\ &\le \int_{Q_R} \Theta(A, B_R(x_0))(x,t)|\nabla w||\nabla w - \nabla v| dxdt.\end{aligned}$$

Here we used the definition of $\Theta(A, B_R(x_0))$ in the last inequality. Using Holder's inequality with exponents $s_1 = \frac{2\theta_1}{\theta_1 - 2}, \theta_1$ and 2 gives

$$\begin{aligned}\Lambda_2 \fint_{Q_R} |\nabla w - \nabla v|^2 \le & \left(\fint_{Q_R} \Theta(A, B_R(x_0))(x,t)^{s_1} dxdt\right)^{1/s_1} \left(\fint_{Q_R} |\nabla w|^{\theta_1} dxdt\right)^{1/\theta_1} \\ & \times \left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2}.\end{aligned}$$

In other words,

$$\left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2} \le \Lambda_2^{-1} [A]_{s_1}^R \left(\fint_{Q_R} |\nabla w|^{\theta_1} dxdt\right)^{1/\theta_1}.$$

After using the inequality (4.7.5) in Theorem 4.7.1 we get (4.7.11). ■

Lemma 4.7.4 *Let θ_1 be the constant in Theorem 4.7.1. There exists a functions $v \in C(t_0 - R^2, t_0; L^2(B_R)) \cap L^2(t_0 - R^2, t_0; H^1(B_R)) \cap L^\infty(t_0 - \frac{1}{4}R^2, t_0; W^{1,\infty}(B_{R/2}))$ such that*

$$||\nabla v||_{L^\infty(Q_{R/2})} \le C \fint_{Q_{2R}} |\nabla u| dxdt + C \frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{4.7.14}$$

and

$$\fint_{Q_R} |\nabla u - \nabla v| dxdt \le C \frac{|\mu|(Q_{2R})}{R^{N+1}} + C[A]_{s_1}^R \left(\fint_{Q_{2R}} |\nabla u| dxdt + \frac{|\mu|(Q_{2R})}{R^{N+1}}\right), \tag{4.7.15}$$

where $s_1 = \frac{2\theta_1}{\theta_1 - 2}$ and $C = C(N, \Lambda_1, \Lambda_2)$.

Proof. Let w and v be in equations (4.7.3) and (4.7.10). By standard interior regularity and inequality (4.7.5) in Theorem 4.7.1 and (4.7.12) in Lemma 4.7.3 we have

$$\begin{aligned}||\nabla v||_{L^\infty(Q_{R/2})} &\le c_1 \left(\fint_{Q_R} |\nabla v|^2 dxdt\right)^{1/2} \\ &\le c_2 \left(\fint_{Q_R} |\nabla w|^2 dxdt\right)^{1/2} \\ &\le c_3 \fint_{Q_{2R}} |\nabla w| dxdt.\end{aligned}$$

Thus, we get (4.7.14) from (4.7.4) in Theorem 4.7.1.
On the other hand, (4.7.11) in Lemma 4.7.3 and Holder's inequality yield

$$\fint_{Q_R} |\nabla w - \nabla v| dxdt \leq c_4 [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt.$$

It leads

$$\fint_{Q_R} |\nabla u - \nabla v| dxdt \leq \fint_{Q_R} |\nabla u - \nabla w| dxdt + c_4 [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt.$$

Consequently, we get (4.7.15) from (4.7.4) in Theorem 4.7.1. The proof is complete. ∎

4.7.2 Boundary Estimates

In this subsection, we focus on the corresponding estimates near the boundary.
Let $x_0 \in \partial\Omega$ be a boundary point and for $R > 0$ and $t_0 \in (-T, T)$, we set $\tilde{\Omega}_{6R} = \tilde{\Omega}_{6R}(x_0, t_0) = (\Omega \cap B_{6R}(x_0)) \times (t_0 - (6R)^2, t_0)$ and $Q_{6R} = Q_{6R}(x_0, t_0)$.
We consider the unique solution w to the equation

$$\begin{cases} w_t - div\,(A(x,t,\nabla w)) = 0 \; in \; \tilde{\Omega}_{6R}, \\ w = u \qquad on \;\; \partial_p \tilde{\Omega}_{6R}. \end{cases} \tag{4.7.16}$$

In what follows we extend μ and u by zero to $(\Omega \times (-\infty, T))^c$ and then extend w by u to $\mathbb{R}^{N+1} \backslash \tilde{\Omega}_{6R}$.

In order to obtain estimates for w as in Theorem 4.7.1 we require the domain Ω to be satisfied $2-$Capacity uniform thickness condition.

4.7.2.1 2-Capacity uniform thickness domain

It is well known that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants $c_0, r_0 > 0$, there exist $p_0 \in (\frac{2N}{N+2}, 2)$ and $C = C(N, c_0) > 0$ such that

$$\mathrm{Cap}_{p_0}(\overline{B_r(x)} \cap (\mathbb{R}^N \backslash \Omega), B_{2r}(x)) \geq C r^{N - p_0}, \tag{4.7.17}$$

for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^N \backslash \Omega$, see [47, 57].

Theorem 4.7.5 *Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let w be in (4.7.16) with $0 < 6R \leq r_0$. There exist constants $\theta_2 > 2$, $\beta_2 \in (0, \frac{1}{2}]$, C_2, C_3 depending on $N, \Lambda_1, \Lambda_2, c_0$ and C_1 depending on N, Λ_1, Λ_2 such that*

$$\fint_{Q_{6R}} |\nabla u - \nabla w| dxdt \leq C_1 \frac{|\mu|(\tilde{\Omega}_{6R})}{R^{N+1}}, \tag{4.7.18}$$

$$\left(\fint_{Q_{\rho/2}(z,s)} |\nabla w|^{\theta_2} dxdt \right)^{\frac{1}{\theta_2}} \leq C_2 \fint_{Q_{3\rho}(z,s)} |\nabla w| dxdt, \tag{4.7.19}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt\right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2} \left(\fint_{Q_{\rho_2}(y,s)} |w|^2 dxdt\right)^{1/2}, \tag{4.7.20}$$

and

$$\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt\right)^{1/2} \le C_3 \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2-1} \left(\fint_{Q_{\rho_2}(z,s)} |\nabla w|^2 dxdt\right)^{1/2}, \tag{4.7.21}$$

for any $Q_{3\rho}(z,s) \subset Q_{6R}$, $y \in \partial\Omega$, $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{6R}$ and $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$

Proof. 1. For $\eta \in C_c^\infty([t_0-(6R)^2,t_0))$, $0 \le \eta \le 1$, $\eta_t \le 0$ and $\eta(t_0-(6R)^2)=1$. Using $\varphi = T_k(u-w)\eta$, for any $k>0$, as a test function for (4.7.1) and (4.7.16), we get

$$\int_{\tilde{\Omega}_{6R}} (u-w)_t T_k(u-w)\eta dxdt$$
$$+\int_{\tilde{\Omega}_{6R}} \left(A(x,t,\nabla u)-A(x,t,\nabla w)\right)\nabla T_k(u-w)\eta dxdt = \int_{\tilde{\Omega}_{6R}} T_k(u-w)\eta d\mu.$$

Thanks to (4.1.3), we obtain

$$-\int_{\tilde{\Omega}_{6R}} \overline{T}_k(u-w)\eta_t dxdt + \Lambda_2 \int_{\tilde{\Omega}_{6R}} |\nabla T_k(u-w)|^2 \eta dxdt \le k|\mu|(\tilde{\Omega}_{6R}),$$

where $\overline{T}_k(s) = \int_0^s T_k(\tau)d\tau$. As in [13, Proposition 2.8], we also verify that

$$|||\nabla(u-w)|||_{L^{\frac{N+2}{N+1},\infty}(\tilde{\Omega}_{6R})} \le c_1|\mu|(\tilde{\Omega}_{6R}).$$

Hence we get (4.7.18).

2. We need to prove that

$$\fint_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \le \frac{1}{2}\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^2 dxdt + c_7 \left(\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^{p_0} dxdt\right)^{\frac{2}{p_0}}, \tag{4.7.22}$$

for all $Q_{\frac{26}{10}r}(z,s) \subset Q_{6R} = Q_{6R}(x_0,t_0)$. Here the constant p_0 is in inequality (4.7.17). Suppose that $B_r(z) \subset \Omega$. Take $\rho \in (0,r]$. Let $\varphi \in C_c^\infty(B_\rho(z))$, $\eta \in C_c^\infty((s-\rho^2,s])$ be such that $0 \le \varphi,\eta \le 1$, $\varphi = 1$ in $B_{\rho/2}(z)$, $\eta = 1$ in $[s-\rho^2/4,s]$ and $|\nabla\varphi| \le c_1/\rho$, $|\eta_t| \le c_1/\rho^2$. We denote

$$\tilde{w}_{B_\rho(z)}(t) = \left(\int_{B_\rho(z)} \varphi(x)^2 dx\right)^{-1} \int_{B_\rho(z)} w(x,t)\varphi(x)^2 dx.$$

Using $\varphi = (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2$ as a test function for the equation (4.7.16) we have for all $s' \in [s-\rho^2/4,s]$

$$\int_{B_\rho(z)\times(s-\rho^2,s')} (w-\tilde{w}_{B_\rho(z)})_t (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt$$
$$+\int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\left((w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2\right) dxdt = 0.$$

Here we used the equality $\int_{B_\rho(z)\times(s-\rho^2,s')} \left(\tilde{w}_{B_\rho(z)}\right)_t (w-\tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt = 0$.
Thus, we can write

$$\begin{aligned}
\frac{1}{2}\int_{B_\rho(z)} &(w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla w\varphi^2\eta^2 dxdt \\
&= -2\int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\varphi\varphi\eta^2(w-\tilde{w}_{B_\rho(z)})dxdt \\
&\quad + \int_{B_\rho(z)\times(s-\rho^2,s')} (w-\tilde{w}_{B_\rho(z)})^2\varphi^2\eta\eta_t dxdt.
\end{aligned}$$

From conditions (4.1.2) and (4.1.3), we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \Lambda_2\int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w|^2\varphi^2\eta^2 dxdt \\
&\le 2\Lambda_1\int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w||\nabla\varphi|\varphi\eta^2|w-\tilde{w}_{B_\rho(z)}|dxdt + \frac{c_8}{\rho^2}\int_{Q_\rho(z,s)} (w-\tilde{w}_{B_\rho(z)})^2 dxdt.
\end{aligned}$$

Using Holder inequality we can verify that

$$\begin{aligned}
\sup_{s'\in[s-\rho^2/4,s]} &\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx \\
&+ \int_{Q_{\rho/2}(z,s)} |\nabla w|^2 dxdt \le \frac{c_9}{\rho^2}\int_{Q_\rho(z,s)} |w-\tilde{w}_{B_\rho(z)}|^2 dxdt. \qquad (4.7.23)
\end{aligned}$$

On the other hand, for any $s'\in[s-\rho^2/4,s]$

$$\int_{B_{\rho/2}(z)} (w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2 dx \le 2(1+2^{N+2})\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx, \quad (4.7.24)$$

where $\varphi_1(x)=\varphi(z+2(x-z))$ for all $x\in B_{\rho/2}(z)$ and

$$\tilde{w}_{B_{\rho/2}(z)} = \left(\int_{B_{\rho/2}(z)} \varphi_1(x)^2 dx\right)^{-1}\int_{B_{\rho/2}(z)} w(x,t)\varphi_1(x)^2 dx.$$

In fact, since $0\le\varphi\le 1$ and $\varphi=1$ in $B_{\rho/2}(z)$ thus

$$\begin{aligned}
\int_{B_{\rho/2}(z)} &(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2 dx \\
&\le 2\int_{B_{\rho/2}(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2 dx + 2^{N+1}(\tilde{w}_{B_{\rho/2}(z)}(s')-\tilde{w}_{B_\rho}(z)(s'))^2|B_{\rho/4}(z)| \\
&\le 2\int_{B_\rho(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + 2^{N+2}\int_{B_{\rho/2}(z)} (w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2\varphi_1^2 dx \\
&\quad +2^{N+2}\int_{B_{\rho/2}(z)} (w(s')-\tilde{w}_{B_\rho(z)}(s'))^2\varphi_1^2 dx.
\end{aligned}$$

which yields (4.7.24) due to the following inequality

$$\int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_{\rho/2}(z)}(s'))^2 \varphi_1^2 dx \leq \int_{B_{\rho/2}(z)} (w(s') - l)^2 \varphi_1^2 dx \quad \forall l \in \mathbb{R}.$$

Therefore,

$$\sup_{s' \in [s-\rho^2/4, s]} \int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_{\rho/2}(z)}(s'))^2 dx + \int_{Q_{\rho/2}(z,s)} |\nabla w|^2 dxdt \leq \frac{c_{10}}{\rho^2} \int_{Q_\rho(z,s)} |w - \tilde{w}_{B_\rho(z)}|^2 dxdt. \tag{4.7.25}$$

Now we use estimate (4.7.25) for $\rho = r/2$, we have

$$\begin{aligned}
\int_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt &\leq \frac{c_{10}}{r^2} \int_{Q_{r/2}(z,s)} (w - \tilde{w}_{B_{r/2}(z)})^2 dxdt \\
&\leq \frac{c_{10}}{r^2} \left(\sup_{s' \in [s-r^2/4, s]} \int_{B_{r/2}(z)} (w(s') - \tilde{w}_{B_{r/2}(z)}(s'))^2 dx \right)^{\frac{2}{N+2}} \\
&\quad \times \int_{s-r^2/4}^{s} \left(\int_{B_{r/2}(z)} (w - \tilde{w}_{B_{r/2}(z)})^2 dx \right)^{\frac{N}{N+2}} dt.
\end{aligned}$$

After we again use estimate (4.7.25) for $\rho = r$ we get

$$\begin{aligned}
\int_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \leq \frac{c_{11}}{r^2} &\left(\frac{1}{r^2} \int_{Q_r(z,s)} |w - \tilde{w}_{B_r(z)}|^2 dxdt \right)^{\frac{2}{N+2}} \\
&\times \int_{s-r^2/4}^{s} \left(\int_{B_{\rho/2}(z)} (w - \tilde{w}_{B_{r/2}(z)})^2 dx \right)^{\frac{N}{N+2}} dt.
\end{aligned}$$

Thanks to a Sobolev-Poincare inequality, we obtain

$$\int_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \leq \frac{c_{12}}{r^2} \left(\int_{Q_r(z,s)} |\nabla w|^2 dxdt \right)^{\frac{2}{N+2}} \int_{Q_{r/2}(z,s)} |\nabla w|^{\frac{2N}{N+2}} dxdt.$$

Since $p_0 \in (\frac{2N}{N+2}, 2)$, thanks to Holder inequality we get (4.7.22).
Finally, we consider the case $B_r(z) \cap \Omega \neq \emptyset$. In this case we choose $z_0 \in \partial\Omega$ such that $|z - z_0| = \operatorname{dist}(z, \partial\Omega)$. Then $|z_0 - z| < r$ and thus $\frac{1}{4}r \leq \rho_1 \leq \frac{1}{2}r$,

$$B_{\frac{1}{4}r}(z) \subset B_{\frac{5}{4}r}(z_0) \subset B_{\rho_1 + r}(z_0) \subset B_{\rho_1 + \frac{11}{10}r}(z_0) \subset B_{\frac{16}{10}r}(z_0) \subset B_{\frac{26}{10}r}(z) \subset B_{6R}(x_0). \tag{4.7.26}$$

Let $\varphi \in C_c^\infty(B_{\rho_1 + \frac{11}{10}r}(z_0))$ be such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_{\rho_1 + r}(z_0)$ and $|\nabla \varphi| \leq C/r$. For $\frac{1}{2}r \leq \rho_2 \leq r$, let $\eta \in C_c^\infty((s - \rho_2^2, s])$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in $[s - \rho_2^2/4, s]$ and $|\eta_t| \leq c/r^2$. Using $\phi = w\varphi^2\eta^2$ as a test function for (4.7.16) we have for any $s' \in (s - \rho_2^2, s)$

$$\int_{(B_{\rho_1 + \frac{11}{10}r}(z_0) \cap \Omega) \times (s - \rho_2^2, s')} w_t w \varphi^2 \eta^2 dxdt + \int_{(B_{\rho_1 + \frac{11}{10}r}(z_0) \cap \Omega) \times (s - \rho_2^2, s')} A(x, t, \nabla w) \nabla \left(w\varphi^2\eta^2 \right) dxdt = 0.$$

As above we also get

$$\sup_{s'\in[s-\rho_2^2/4,s]}\int_{B_{\rho_1+r}(z_0)} w^2(s')dx + \int_{B_{\rho_1+r}(z_0)\times(s-\rho_2^2/4,s)} |\nabla w|^2 dxdt \le \frac{c_{13}}{r^2}\int_{B_{\rho_1+\frac{11}{10}r}(z_0)\times(s-\rho_2^2,s)} w^2 dxdt.$$

In particular, for $\rho_1 = \frac{1}{4}r$, $\rho_2 = \frac{1}{2}r$ and using (4.7.26) yield

$$\int_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt \le \frac{c_{14}}{r^2}\int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2/4,s)} w^2 dxdt, \tag{4.7.27}$$

and $\rho_1 = (\frac{1}{4}+\frac{1}{10})r, \rho_2 = r$,

$$\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \le \frac{c_{15}}{r^2}\int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2,s)} w^2 dxdt.$$

Set $K_1 = \{w=0\}\cap \overline{B}_{\frac{29}{20}r}(z_0)$ and $K_2 = \{w=0\}\cap \overline{B}_{\frac{1}{4}r+\frac{11}{10}r}(z_0)$, Since $\mathbb{R}^N\backslash\Omega$ satisfies an uniformly $2-$thick,we have the following estimates

$$\mathrm{Cap}_2(K_1, B_{\frac{29}{10}r}(z_0)) \ge c_{16}r^{N-2} \quad\text{and}\quad \mathrm{Cap}_{p_0}(K_2, B_{\frac{1}{2}r+\frac{11}{5}r}(z_0)) \ge c_{16}r^{N-p_0}.$$

So, by Sobolev-Poincare's inequality we get

$$\fint_{B_{\frac{29}{20}r}(z_0)} w^2 dx \le c_{17}r^2 \fint_{B_{\frac{5}{2}r}(z)} |\nabla w|^2 dx, \tag{4.7.28}$$

and

$$\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2 dxdt \le c_{18}r^2\left(\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} |\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}} \le c_{19}r^2\left(\fint_{B_{\frac{5}{2}r}(z_0)} |\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}}.$$

Leads to

$$\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \le c_{20}\int_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^2 dxdt, \tag{4.7.29}$$

and

$$\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx \le c_{21}r^{N+2}\left(\fint_{B_{\frac{5}{2}r}(z_0)} |\nabla w|^{p_0}(t)dx\right)^{\frac{2}{p_0}}. \tag{4.7.30}$$

From (4.7.27), we have

$$\begin{aligned}\fint_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt &\le \frac{c_{22}}{r^{N+4}}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)\times(s-r^2/4,s)} w^2 dxdt\\ &\le \frac{c_{22}}{r^{N+4}}\left(\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx\right)^{1-\frac{p_0}{2}}\int_{s-r^2/4}^{s}\left(\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx\right)^{\frac{p_0}{2}} dt.\end{aligned}$$

Using (4.7.30), (4.7.29) and Holder's inequality we get

$$\begin{aligned}\fint_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt &\leq \frac{c_{23}}{r^{N+4}} \left(\int_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^2 dxdt \right)^{1-\frac{p_0}{2}} r^{\frac{N+2}{2}p_0 - N} \int_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^{p_0} dxdt \\ &= c_{24} \left(\fint_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^2 dxdt \right)^{1-\frac{p_0}{2}} \fint_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^{p_0} dxdt \\ &\leq \frac{1}{2} \fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^2 dxdt + c_{25} \left(\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^{p_0} dxdt \right)^{\frac{2}{p_0}}.\end{aligned}$$

So we proved (4.7.22).
Therefore, By Gehring's Lemma (see [60]) we get (4.7.19).
3. Now we prove (4.7.20). Let $y \in \partial\Omega$, $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{6R}$ with $\rho_1 \leq \rho_2/4$. First, we will show that there exists a constant $\beta_2 = \beta_2(N, \Lambda_1, \Lambda_2, c_0) \in (0, 1/2]$ such that

$$\operatorname{osc}(w, Q_{\rho_1}(y,s)) \leq c_{26} \left(\frac{\rho_1}{\rho_2} \right)^{\beta_2} \operatorname{osc}(w, Q_{\rho_2/2}(y,s)), \tag{4.7.31}$$

where $\operatorname{osc}(w, A) = \sup_A w - \inf_A w$.
Indeed, since

$$\int_0^1 \frac{\operatorname{Cap}_{1,2}(\Omega^c \cap B_r(z), B_{2r}(z))}{r^{N-2}} \frac{dr}{r} = +\infty \quad \forall z \in \partial\Omega.$$

thus by the Wiener criterion (see [83]), we have w is continuous up to $\partial_p \tilde{\Omega}_{6R}$. So, we can choose $\varphi = (V - M_{4\rho_1})\eta^2 \in L^2(-\infty, T; H_0^1(\Omega \cap B_{6R}(x_0)))$ as test function in (4.7.16), where
a. $\eta \in C^\infty(Q_{4\rho_1}(y,s))$, $0 \leq \eta \leq 1$ such that $\eta = 1$ in $Q_{\rho_1/2}(y, s - \frac{17}{4}\rho_1^2)$, $\operatorname{supp}(\eta) \subset\subset Q_{\rho_1}(y, s - 4\rho_1^2)$ and $|\nabla \eta| \leq c_{27}/\rho_1$, $|\eta_t| \leq c_{28}/\rho_1^2$.
b. $M_{4\rho_1} = \sup_{Q_{4\rho_1}(y,s)} w$ and $V = \inf\{M_{4\rho_1} - w, M_{4\rho_1}\}$ in $\tilde{\Omega}_{6R}$, $V = M_{4\rho_1}$ outside $\tilde{\Omega}_{6R}$.
We have

$$\begin{aligned}&\int_{\tilde{\Omega}_{6R}} w_t (V - M_{4\rho_1}) \eta^2 dxdt \\ &\quad + \int_{\tilde{\Omega}_{6R}} 2\eta A(x,t,\nabla w)\nabla\eta (V - M_{4\rho_1}) dxdt + \int_{\tilde{\Omega}_{6R}} \eta^2 A(x,t,\nabla w)\nabla V dxdt = 0,\end{aligned}$$

which implies

$$\begin{aligned}\int_{\tilde{\Omega}_{6R}} \eta^2 A(x,t,-\nabla V)(-\nabla V) dxdt &= \int_{\tilde{\Omega}_{6R}} 2\eta A(x,t,-\nabla V)\nabla\eta (V - M_{4\rho_1}) dxdt \\ &\quad - \int_{\tilde{\Omega}_{6R}} (V - M_{4\rho_1})_t (V - M_{4\rho_1}) \eta^2 dxdt.\end{aligned}$$

Using (4.1.2) and (4.1.3) we get

$$\begin{aligned}
&\Lambda_2 \int_{\tilde{\Omega}_{6R}} \eta^2 |\nabla V|^2 dxdt \\
&\quad \le 2\Lambda_1 \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta||V - M_{4\rho_1}| dxdt - 1/2 \int_{\tilde{\Omega}_{6R}} \left((V - M_{4\rho_1})^2 - M_{4\rho_1}^2 \right) (\eta^2)_t dxdt \\
&\quad \le 2\Lambda_1 M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt + 2M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta V |\eta_t| dxdt.
\end{aligned}$$

Since $\mathrm{supp}(|\nabla V|) \cap \mathrm{supp}(\eta) \subset \tilde{\Omega}_{6R}$, thus

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt &\le c_{29} M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla \eta| dxdt + \int_{\mathbb{R}^{N+1}} V \left(\eta |\eta_t| + |\nabla \eta|^2 \right) dxdt \right) \\
&\le c_{30} M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla \eta| dxdt + \frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V dxdt \right).
\end{aligned} \tag{4.7.32}$$

By [50, Theorem 6.31, p. 132], for any $\sigma \in (0, 1 + 2/N)$ there holds

$$\left(\fint_{Q_{\rho_1}(y,s-4\rho_1^2)} V^\sigma dxdt \right)^{1/\sigma} \le c_{31} \inf_{Q_{\rho_1}(y,s)} V = c_{31}(M_{4\rho_1} - \sup_{Q_{\rho_1}(y,s)} w) = c_{31}(M_{4\rho_1} - M_{\rho_1}). \tag{4.7.33}$$

In particular,

$$\frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V dxdt \le c_{32} \rho_1^N (M_{4\rho_1} - M_{\rho_1}). \tag{4.7.34}$$

We need to estimate $\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt$. Using Holder inequality and (4.7.33), for $\varepsilon \in (0, \min\{2/N, 1\})$ we have

$$\begin{aligned}
\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt &\le \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \left(\int_{\tilde{\Omega}_{6R}} V^{1+\varepsilon} |\nabla \eta|^2 dxdt \right)^{1/2} \\
&\le c_{28} \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \left(\int_{Q_{\rho_1}(y,s-4\rho_1^2)} V^{1+\varepsilon} dxdt \right)^{1/2} \\
&\le c_{33} \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \rho_1^{N/2} (M_{4\rho_1} - M_{\rho_1})^{(1+\varepsilon)/2}.
\end{aligned}$$

To estimate $\left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2}$, we can choose $\varphi = ((V + \delta)^{-\varepsilon} - (M_{4\rho_1} + \delta)^{-\varepsilon})\eta^2$, for $\delta > 0$, as test function in (4.7.16), we will get

$$\begin{aligned}
&\int_{\tilde{\Omega}_{6R}} \eta^2 (V + \delta)^{-(1+\varepsilon)} |\nabla V|^2 dxdt \\
&\qquad \le c_{34} \int_{\tilde{\Omega}_{6R}} \eta (V + \delta)^{-\varepsilon} |\nabla V||\nabla \eta| dxdt + c_{34} \int_{\tilde{\Omega}_{6R}} \eta (V + \delta)^{1-\varepsilon} |\eta_t| dxdt.
\end{aligned}$$

Thanks to Holder's inequality, we obtain

$$\begin{aligned}\int_{\tilde{\Omega}_{6R}} \eta^2 (V+\delta)^{-(1+\varepsilon)} |\nabla V|^2 dxdt &\leq c_{35} \int_{\tilde{\Omega}_{6R}} (V+\delta)^{1-\varepsilon} \left(\eta|\eta_t| + |\nabla \eta|^2\right) dxdt \\ &\leq c_{36}\rho_1^2 \int_{Q_{\rho_1}(y,s-4\rho_1^2)} (V+\delta)^{1-\varepsilon} dxdt.\end{aligned}$$

Letting $\delta \to 0$ and using (4.7.33), we get

$$\begin{aligned}\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt &\leq c_{36}\rho_1^2 \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V^{1-\varepsilon} dxdt \\ &\leq c_{37}\rho_1^N \left(M_{4\rho_1} - M_{\rho_1}\right)^{1-\varepsilon}.\end{aligned}$$

Thus,

$$\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt \leq c_{38}\rho_1^N (M_{4\rho_1} - M_{\rho_1}).$$

Combining this with (4.7.32) and (4.7.34),

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt \leq c_{39}\rho_1^N M_{4\rho_1} \left(M_{4\rho_1} - M_{\rho_1}\right).$$

Note that $\eta V = M_{4\rho_1}$ in $\left(\Omega^c \cap B_{\rho_1/2}(y)\right) \times (s - \frac{9}{2}\rho_1^2, s - \frac{17}{4}\rho_1^2)$ thus

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt &\geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} \int_{\mathbb{R}^{\mathbb{N}}} |\nabla(\eta V)|^2 dxdt \\ &\geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} M_{4\rho_1}^2 \mathrm{Cap}_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) dt \\ &\geq c_{40} M_{4\rho_1}^2 \rho_1^N.\end{aligned}$$

Here we used $Cap_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) \geq c\rho_1^{N-2}$ in the last inequality. It follows

$$M_{4\rho_1} \leq c_{41}(M_{4\rho_1} - M_{\rho_1}).$$

So

$$\sup_{Q_{\rho_1}(y,s)} w \leq \gamma \sup_{Q_{4\rho_1}(y,s)} w \quad \text{where} \quad \gamma = \frac{c_{41}}{c_{41}+1} < 1.$$

Of course, above estimate is also true when we replace w by $-w$. These give,

$$\mathrm{osc}(w, Q_{\rho_1}(y,s)) \leq \gamma \mathrm{osc}(w, Q_{4\rho_1}(y,s)).$$

It follows (4.7.31).
We come back the proof of (4.7.20).
Since $w = 0$ outside Ω_T this leads to

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt\right)^{1/2} \leq c_{42} \mathrm{osc}(w, Q_{\rho_2/2}(y,s)).$$

On the other hand, By [50, Theorem 6.30, p. 132] we have

$$\sup_{Q_{\rho_2/2}(y,s)} w \leq c_{43} \left(\fint_{Q_{\rho_2}(y,s)} (w^+)^2 dxdt \right)^{1/2} \quad \text{and}$$
$$\sup_{Q_{\rho_2/2}(y,s)} (-w) \leq c_{44} \left(\fint_{Q_{\rho_2}(y,s)} (w^-)^2 dxdt \right)^{1/2}.$$

Thus, we get (4.7.20).
Next, we have (4.7.21) for case $z = y \in \partial\Omega$ since from Caccippoli's inequality,

$$\int_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt \leq \frac{c_{45}}{\rho_1^2} \int_{Q_{2\rho_1}(z,s)} |w|^2 dxdt,$$

and using Sobolev-Poincare's inequality as in (4.7.28),

$$\int_{Q_{\rho_2}(z,s)} |w|^2 dxdt \leq c_{46} \rho_2^2 \int_{Q_{\rho_2}(z,s)} |\nabla w|^2 dxdt.$$

We now prove (4.7.21). Take $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$, it is enough to consider the case $\rho_1 \leq \rho_2/20$. Clearly, if $B_{\rho_2/4}(z) \subset \Omega$ then (4.7.21) follows from (4.7.7) in Theorem 4.7.1. We consider $B_{\rho_2/4}(z) \cap \partial\Omega \neq \emptyset$, let $z_0 \in B_{\rho_2/4}(z) \cap \partial\Omega$ such that $|z - z_0| = \text{dist}(z, \partial\Omega) \leq \rho_2/4$. Obviously, if $\rho_1 < |z - z_0|/4$ and $z \notin \Omega$, then (4.7.21) is trivial. If $\rho_1 < |z - z_0|/4$ and $z \in \Omega$, then (4.7.21) follows from (4.7.7) in Theorem 4.7.1.
Now assume $\rho_1 \geq |z - z_0|/4$ then since $Q_{\rho_1}(z,s) \subset Q_{5\rho_1}(z_0,s)$

$$\begin{aligned}
\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt \right)^{1/2} &\leq c_{47} \left(\fint_{Q_{5\rho_1}(z_0,s)} |\nabla w|^2 dxdt \right)^{1/2} \\
&\leq c_{48} \left(\frac{\rho_1}{\rho_2} \right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2/4}(z_0,s)} |\nabla w|^2 dxdt \right)^{1/2} \\
&\leq c_{49} \left(\frac{\rho_1}{\rho_2} \right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2/2}(z,s)} |\nabla w|^2 dxdt \right)^{1/2},
\end{aligned}$$

which implies (4.7.21). ■

Corollary 4.7.6 *Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let β_2 be the constant in Theorem 4.7.5. For $2 - \beta_2 < \theta < N + 2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, \theta) > 0$ such that for any $B_\rho(y) \cap \partial\Omega \neq \emptyset$, $s \in (-T, T)$, $0 < \rho \leq r_0$*

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \leq C \rho^{N+3-\theta} \left(\left(\frac{T_0}{r_0} \right)^{N+3-\theta} + 1 \right) ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega \times (-T,T))}, \tag{4.7.35}$$

where $T_0 = diam(\Omega) + T^{1/2}$.

Proof. Take $B_{\rho_2/4}(y)\cap\partial\Omega \neq \emptyset$ and $s \in (-T,T)$, $\rho_2 \leq 2r_0$. Let $y_0 \in B_{\rho_2/4}(y)\cap\partial\Omega$ such that $|y-y_0| = dist(y,\partial\Omega) \leq \rho_2/4$, thus $Q_{\rho_2/4}(y,s) \subset Q_{\rho_2/2}(y_0,s)$ For any $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s)$ with $\rho_1 \leq \rho_2/4$, we take w as in Theorem 4.7.5 with $Q_{6R} = Q_{\rho_2/2}(y_0,s)$. Thus,

$$\int_{Q_{\rho_1}(y,s)} |\nabla w| dxdt \leq c_1 \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2/4}(y,s)} |\nabla w| dxdt,$$
$$\int_{Q_{\rho_2/2}(y_0,s)} |\nabla u - \nabla w| dxdt \leq c_2\rho_2|\mu|(Q_{\rho_2/2}(y_0,s)).$$

As in the proof of Corollary 4.7.2, we get the result. ■

4.7.2.2 Reifenberg flat domain

In this subsection, we always assume that A satisfies (4.2.27). Also, we assume that Ω is a (δ,R_0)- Reifenberg flat domain with $0<\delta<1/2$. Fix $x_0 \in \partial\Omega$ and $0<R<R_0/6$. We have a density estimate

$$|B_t(x)\cap(\mathbb{R}^N\backslash\Omega)| \geq c|B_t(x)| \;\forall x\in\partial\Omega, 0<t<R_0, \tag{4.7.36}$$

with $c = ((1-\delta)/2)^N \geq 4^{-N}$.
In particular, $\mathbb{R}^N\backslash\Omega$ satisfies uniformly 2−thick with constants $c, r_0 = R_0$.
Next we set $\rho = R(1-\delta)$ so that $0<\rho/(1-\delta)<R_0/6$. By the definition of Reifenberg flat domains, there exists a coordinate system $\{y_1,y_2,...,y_N\}$ with the origin $0\in\Omega$ such that in this coordinate system $x_0 = (0,...,0,-\rho\delta/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega\cap B_\rho(0) \subset B_\rho(0)\cap\{y=(y_1,y_2,....,y_N): y_N > -2\rho\delta/(1-\delta)\}.$$

Since $\delta<1/2$ we have

$$B_\rho^+(0) \subset \Omega\cap B_\rho(0) \subset B_\rho(0)\cap\{y=(y_1,y_2,....,y_N): y_N > -4\rho\delta\},$$

where $B_\rho^+(0) := B_\rho(0)\cap\{y=(y_1,y_2,...,y_N): y_N>0\}$.
Furthermore we consider the unique solution

$$v \in C(t_0-\rho^2,t_0;L^2(\Omega\cap B_\rho(0)))\cap L^2(t_0-\rho^2,t_0;H^1(\Omega\cap B_\rho(0))) \tag{4.7.37}$$

to the following equation

$$\begin{cases} v_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla v)\right) = 0 \;in\; \tilde{\Omega}_\rho(0), \\ v = w \qquad on \;\; \partial_p\tilde{\Omega}_\rho(0), \end{cases} \tag{4.7.38}$$

where $\tilde{\Omega}_\rho(0) = (\Omega\cap B_\rho(0))\times(t_0-\rho^2,t_0)$ $(-T<t_0<T)$.
We put $v=w$ outside $\tilde{\Omega}_\rho(0)$. As Lemma 4.7.3 we have the following Lemma.

Lemma 4.7.7 *Let θ_2 be the constant in Theorem 4.7.5. There exists constants $C_1 = C_1(N,\Lambda_1,\Lambda_2)$, $C_2 = C_2(\Lambda_1,\Lambda_2)$ such that*

$$\left(\fint_{Q_\rho(0,t_0)} |\nabla w - \nabla v|^2\right)^{1/2} \leq [A]_{s_2}^R \fint_{Q_\rho(0,t_0)} |\nabla w| dxdt, \tag{4.7.39}$$

with $s_2 = \frac{2\theta_2}{\theta_2 - 2}$ *and*

$$C_2^{-1}\int_{Q_\rho(0,t_0)}|\nabla v|^2dxdt \le \int_{Q_\rho(0,t_0)}|\nabla w|^2dxdt \le C_2\int_{Q_\rho(0,t_0)}|\nabla v|^2dxdt. \tag{4.7.40}$$

We can see that if the boundary of Ω is bad enough, then the L^∞-norm of ∇v up to $\partial\Omega \cap B_\rho(0)\times(t_0-\rho^2,t_0)$ could be unbounded. For our purpose, we will consider another equation :

$$\begin{cases} V_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla V)\right) = 0 \text{ in } Q_\rho^+(0,t_0), \\ V = 0 \quad \text{ on } T_\rho(0,t_0), \end{cases} \tag{4.7.41}$$

where $Q_\rho^+(0,t_0) = B_\rho^+(0)\times(t_0-\rho^2,t_0)$ and $T_\rho(0,t_0) = Q_\rho(0,t_0)\cap\{x_N = 0\}$.
A weak solution V of above problem is understood in the following sense : the zero extension of V to $Q_\rho(0,t_0)$ is in $V \in C(t_0-\rho^2,t_0;L^2(B_\rho(0)))\cap L^2_{\text{loc}}(t_0-\rho^2,t_0;H^1(B_\rho(0)))$ and for every $\varphi \in C_c^1(Q_\rho^+(0,t_0))$ there holds

$$-\int_{Q_\rho^+(0,t_0)} V\varphi_t dxdt + \int_{Q_\rho^+(0,t_0)}\overline{A}_{B_\rho(0)}(t,\nabla V)\nabla\varphi dxdt = 0.$$

We have the following gradient L^∞ estimate up to the boundary for V.

Lemma 4.7.8 (see [48, 49]) *For any weak solution* $V \in C(t_0-\rho^2,t_0;L^2(B_\rho^+(0)))\cap L^2_{loc}(t_0-\rho^2,t_0;H^1(B_\rho^+(0)))$ *of* (4.7.41)*, we have*

$$||\nabla V||_{L^\infty(Q^+_{\rho'/2}(0,t_0))} \le C \fint_{Q^+_{\rho'}(0,t_0)}|\nabla V|^2dxdt \quad \forall\ 0<\rho'\le\rho. \tag{4.7.42}$$

for some constant $C = C(N,\Lambda_1,\Lambda_2) > 0$*. Moreover,* ∇V *is continuous up to* $T_\rho(0,t_0)$*.*

Lemma 4.7.9 *If* $V \in C(t_0-\rho^2,t_0;L^2(B_\rho^+(0)))\cap L^2(t_0-\rho^2,t_0;H^1(B_\rho^+(0)))$ *is a weak solution of* (4.7.41)*, then its zero extension from* $Q_\rho^+(0,t_0)$ *to* $Q_\rho(0,t_0)$ *solves*

$$V_t - div\left(\overline{A}_{B_\rho(0)}(t,\nabla V)\right) = \frac{\partial F}{\partial x_N}, \tag{4.7.43}$$

weakly in $Q_\rho(0,t_0)$*, for* $(x,t) = (x',x_N,t)\in Q_\rho(0,t_0)$*,*
$\overline{A}_{B_\rho(0)} = (\overline{A}^1_{B_\rho(0)},\overline{A}^2_{B_\rho(0)},...,\overline{A}^N_{B_\rho(0)})$*, and* $F(x,t) = \chi_{x_N<0}\overline{A}^N_{B_\rho(0)}(t,\nabla V(x',0,t))$*.*

Proof. Let $g \in C^\infty(\mathbb{R})$ with $g = 0$ on $(-\infty,1/2)$ and $g = 1$ on $(1,\infty)$. Then, for any $\varphi \in C_c^\infty(Q_\rho(0,t_0))$ and $n\in\mathbb{N}$. We have $\varphi_n(x,t) = \varphi_n(x',x_N,t) = g(nx_N)\varphi(x,t) \in C_c^\infty(Q_\rho^+(0,t_0)$. Thus, we get

$$\int_{Q_\rho^+(0,t_0)} V_t\varphi_n dxdt + \int_{Q_\rho^+(0,t_0}\overline{A}_{B_\rho(0)}(t,\nabla V)\nabla\left(g(nx_N)\varphi(x,t)\right)dxdt = 0,$$

which implies

$$\int_{Q_\rho^+(0,t_0)} V_t\varphi_n dxdt + \int_{Q_\rho^+(0,t_0)}\overline{A}_{B_\rho(0)}(t,\nabla V)\nabla\varphi(x,t)g(nx_N)dxdt$$
$$= -\int_0^\rho G(x_N)g'(nx_N)ndx_N.$$

where

$$G(x_N) = \int_{t_0-\rho^2}^{t_0} \int_{|x'|<\sqrt{\rho^2-x_N^2}} \overline{A}^N_{B_\rho(0)}(t,\nabla V)\varphi(x',x_N,t)dx'dt \in C([0,\infty)).$$

Letting $n \to \infty$ we get

$$\int_{Q^+_\rho(0,t_0)} V_t\varphi dxdt + \int_{Q^+_\rho(0,t_0)} \overline{A}_{B_\rho(0)}(t,\nabla V)\nabla\varphi(x,t)dxdt = -G(0)$$
$$= -\int_{Q_\rho(0,t_0)} F\frac{\partial\varphi}{\partial x_N}dxdt.$$

Since $\nabla V = 0, V = 0$ outside Q^+_ρ, therefore we get the result. ■

We now consider a scaled version of equation (4.7.38)

$$\begin{cases} v_t - div\left(\overline{A}_{B_1(0)}(t,\nabla v)\right) = 0 \quad \text{in } \tilde{\Omega}_1(0), \\ v = 0 \quad \text{on } \partial_p\tilde{\Omega}_1(0)\setminus(\Omega\times(-T,T)), \end{cases} \tag{4.7.44}$$

under assumption

$$B^+_1(0) \subset \Omega\cap B_1(0) \subset B_1(0)\cap\{x_N > -4\delta\}. \tag{4.7.45}$$

Lemma 4.7.10 *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N,\Lambda_1,\Lambda_2,\varepsilon) > 0$ such that if $v \in C(t_0-1,t_0;L^2(\Omega\cap B_1(0)))\cap L^2(t_0-1,t_0;H^1(\Omega\cap B_1(0)))$ is a solution of* (4.7.44) *and* (4.7.45) *is satisfied and the bounded*

$$⨍_{Q_1(0,t_0)} |\nabla v|^2 dxdt \le 1, \tag{4.7.46}$$

then there exists a weak solution $V \in C(t_0-1,t_0;L^2(B^+_1(0)))\cap L^2(t_0-1,t_0;H^1(B^+_1(0)))$ of (4.7.41) *with $\rho = 1$, whose zero extension to $Q_1(0,t_0)$ satisfies*

$$⨍_{Q_1(0,t_0)} |v-V|^2 dxdt \le \varepsilon^2, \tag{4.7.47}$$

Proof. We argue by contradiction. Suppose that the conclusion were false. Then, there exist a constant $\varepsilon_0 > 0$, $t_0 \in \mathbb{R}$ and a sequence of nonlinearities $\{A_k\}$ satisfying (4.1.2) and (4.2.27), a sequence of domains $\{\Omega^k\}$, and a sequence of functions $\{v_k\} \subset C(t_0-1,t_0;L^2(\Omega^k\cap B_1(0)))\cap L^2(t_0-1,t_0;H^1(\Omega^k\cap B_1(0)))$ such that

$$B^+_1(0) \subset \Omega^k\cap B_1(0) \subset B_1(0)\cap\{x_N > -1/2k\}, \tag{4.7.48}$$

$$\begin{cases} (v_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla v_k)\right) = 0 \quad \text{in } \tilde{\Omega}^k_1(0), \\ v_k = 0 \quad \text{on } \quad (\partial_p\tilde{\Omega}^k_1(0))\setminus(\Omega^k\times(-T,T)), \end{cases} \tag{4.7.49}$$

and the zero extension of each v_k to $Q_1(0,t_0)$ satisfies

$$⨍_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \le 1 \quad \text{but} \tag{4.7.50}$$

$$⨍_{Q_1(0,t_0)} |v_k - V_k|^2 dxdt \ge \varepsilon_0^2, \tag{4.7.51}$$

for any weak solution V_k of

$$\begin{cases} (V_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla V_k)\right) = 0, \text{ in } Q_1^+(0,t_0), \\ V_k = 0 \text{ on } T_1(0,t_0). \end{cases} \tag{4.7.52}$$

By (4.7.48) and (4.7.50) and Poincare's inequality it following that

$$||v_k||_{L^2(t_0-1,t_0;H^1(B_1(0)))} \le c_1||\nabla v_k||_{L^2(Q_1(0,t_0)} \le c_2,$$

and

$$\begin{aligned} ||(v_k)_t||_{L^2(t_0-1,t_0;H^{-1}(B_1(0)))} &= ||\overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)||_{L^2(t_0-1,t_0;H^{-1}(B_1(0)))} \\ &\le \int_{Q_1(0,t_0)} |\overline{A}_{k,B_1(0)}(t,\nabla v_k)|^2 dxdt \\ &\le c_3 \int_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \\ &\le c_4. \end{aligned}$$

Therefore, using Aubin$-$Lions Lemma, one can find v_0 and a subsequence, still denoted by $\{v_k\}$ such that

$$v_k \to v_0 \text{ weakly in } L^2(t_0-1,t_0,H^1(B_1(0))) \text{ and strongly in } L^2(t_0-1,t_0,L^2(B_1(0))),$$

and

$$(v_k)_t \to (v_0)_t \quad \text{weakly in } L^2(t_0-1,t_0,H^{-1}(B_1(0))).$$

Moreover, $v_0 = 0$ in $Q_1^-(0,t_0) := (B_1(0)\cap\{x_N<0\})\times(1-t_0,1)$ since $v_k = 0$ on outside $\Omega^k \cap Q_1(0,t_0)$ for all k.
To get a contradiction we take V_k to be the unique solution of $(V_k)_t - div\left(\overline{A}_{k,B_1(0)}(t,\nabla V_k)\right) = 0$ in $Q_1^+(0,t_0)$ and $V_k - v_0 \in L^2(t_0-1,t_0,H_0^1(B_1^+(0)))$ and $V_k(t_0-1) = v_0(t_0-1)$. As above, one can find V_0 and a subsequence, still denoted by $\{V_k\}$ such that

$$V_k \to V_0 \text{ weakly in } L^2(t_0-1,t_0,H^1(B_1(0))) \text{ and strongly in } L^2(t_0-1,t_0,L^2(B_1(0))),$$

and

$$(V_k)_t \to (V_0)_t \quad \text{weakly in } L^2(t_0-1,t_0,H^{-1}(B_1)),$$

for some $V_0 \in v_0 + L^2(t_0-1,t_0,H_0^1(B_1^+(0))$ and $V_0(t_0-1) = v_0(t_0-1)$.
Thanks to (4.7.51), the proof would be complete if we could show that $v_0 = V_0$. In fact, Let $\mathcal{J}_k : X \to L^2(Q_1^+(0,t_0),\mathbb{R}^N)$ determined by

$$\mathcal{J}_k(\phi(x,t)) = \overline{A}_{k,B_1(0)}(t,\nabla\phi(x,t)) \text{ for any } \phi \in X,$$

where $X \subset L^2(t_0-1,t_0,H^1(B_1(0)))$ is closures (in the strong topology of $L^2(t_0-1,t_0,H^1(B_1(0))))$ of convex combinations of $\{v_k\}_{k\ge1}\cup\{V_k\}_{k\ge1}\cup\{0\}$.
Since v_k, V_k converge weakly to v_0, V_0 in $L^2(t_0-1,t_0,H^1(B_1(0)))$ resp., thus by Mazur Theorem, X is compact subset of $L^2(t_0-1,t_0,H^1(B_1(0)))$ and $v_0, V_0 \in X$.
Thanks to (4.1.2) and (4.2.27), we get $\mathcal{J}_k(0) = 0$ and

$$||\mathcal{J}_k(\phi_1) - \mathcal{J}_k(\phi_2)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)} \le \Lambda_1||\phi_1-\phi_2||_{L^2(t_0-1,t_0,H^1(B_1(0)))},$$

for every $\phi_1, \phi_2 \in X$ and $k \in \mathbb{N}$. Thus, by Ascoli Theorem, there exist $\mathcal{J} \in C(X, L^2(Q_1^+(0,t_0), \mathbb{R}^N))$ and a subsequence of $\{\mathcal{J}_k\}$, still denote by it, such that

$$\sup_{\phi \in X} ||\mathcal{J}_k(\phi) - \mathcal{J}(\phi)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)} \to 0 \quad \text{as} \quad k \to \infty, \tag{4.7.53}$$

and also for any $\phi_1, \phi_2 \in X$,

$$\int_{Q_1^+(0,t_0)} (\mathcal{J}(\phi_1) - \mathcal{J}(\phi_2)) \,.\, (\nabla\phi_1 - \nabla\phi_2) \, dxdt \geq \Lambda_2 |||\nabla\phi_1 - \nabla\phi_2|||_{L^2(Q_1^+(0,t_0))}. \tag{4.7.54}$$

From (4.7.48), we deduce

$$\int_{Q_1^+(0,t_0)} (v_k - V_k)_t (v_0 - V_0) dxdt \\ + \int_{Q_1^+(0,t_0)} \left(\overline{A}_{k,B_1(0)}(t, \nabla v_k) - \overline{A}_{k,B_1(0)}(t, \nabla V_k)\right) . \nabla(v_0 - V_0) dxdt = 0.$$

We have

$$\int_{Q_1^+(0,t_0)} |\overline{A}_{k,B_1(0)}(\nabla v_k)|^2 dxdt \leq c_9 \int_{Q_1^+(0,t_0)} |\nabla v_k|^2 dxdt \leq c_{10} \quad \text{and}$$
$$\int_{Q_1^+(0,t_0)} |\overline{A}_{k,B_1(0)}(\nabla V_k)|^2 dxdt \leq c_9 \int_{Q_1^+(0,t_0)} |\nabla V_k|^2 dxdt \leq c_{11}.$$

for every k.
Thus there exists a subsequence, still denoted by $\{\overline{A}_{k,B_1(0)}(t, \nabla v_k), \overline{A}_{k,B_1(0)}(t, \nabla V_k)\}$ and a vector field A_1, A_2 belonging to $L^2(Q_1^+(0,t_0), \mathbb{R}^N)$ such that

$$\overline{A}_{k,B_1(0)}(t, \nabla v_k) \to A_1 \quad \text{and} \quad \overline{A}_{k,B_1(0)}(t, \nabla V_k) \to A_2,$$

weakly in $L^2(Q_1^+(0,t_0), \mathbb{R}^N)$. It follows

$$\int_{Q_1^+(0,t_0)} (v_0 - V_0)_t (v_0 - V_0) dxdt + \int_{Q_1^+(0,t_0)} (A_1 - A_2).\nabla(v_0 - V_0) dxdt = 0.$$

Since

$$\int_{Q_1^+(0,t_0)} (v_0 - V_0)_t (v_0 - V_0) dxdt = \int_{B_1^+(0)} (v_0 - V_0)^2(t_0) dx \geq 0,$$

we get

$$\int_{Q_1^+(0,t_0)} (A_1 - A_2).\nabla(v_0 - V_0) dxdt \leq 0. \tag{4.7.55}$$

For our purpose, we need to show that

$$\int_{Q_1^+(0,t_0)} (A_1 - \mathcal{J}(v_0)).\nabla(v_0 - V_0) dxdt \geq 0 \quad \text{and} \tag{4.7.56}$$

$$\int_{Q_1^+(0,t_0)} (A_2 - \mathcal{J}(V_0)).\nabla(V_0 - v_0) dxdt \geq 0. \tag{4.7.57}$$

To do this, we fix a function $g \in X$ and any $\varphi \in C_c^1(Q_1^+(0,t_0))$ such that $\varphi \geq 0$. We have

$$\begin{aligned}
0 &\leq \int_{Q_1^+(0,t_0)} \varphi \left(\overline{A}_{k,B_1(0)}(t,\nabla v_k) - \overline{A}_{k,B_1(0)}(t,\nabla g)\right)(\nabla v_k - \nabla g)\, dxdt \\
&= \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla v_k)\nabla v_k dxdt - \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla v_k)\nabla g dxdt \\
&- \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1(0)}(t,\nabla g)\,(\nabla v_k - \nabla g)\, dxdt \\
&:= B_1 + B_2 + B_3.
\end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty} B_2 = -\int_{Q_1^+(0,t_0)} \varphi A_1 \nabla g dxdt \quad \text{and} \quad \lim_{k\to\infty} B_3 = -\int_{Q_1^+(0,t_0)} \varphi \mathcal{J}(g)\,(\nabla v_0 - \nabla g)\, dxdt.$$

Moreover, we have

$$\begin{aligned}
B_1 &= -\int_{Q_1^+(0,t_0)} (v_k)_t \varphi v_k dxdt - \int_{Q_1^+(0,t_0)} \overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)\nabla\varphi v_k dxdt \\
&= \frac{1}{2}\int_{Q_1^+(0,t_0)} v_k^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} \overline{A}_{k,Q_1(0,t_0)}(\nabla v_k)\nabla\varphi v_k dxdt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} B_1 &= \frac{1}{2}\int_{Q_1^+(0,t_0)} v_0^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla\varphi v_0 dxdt \\
&= -\int_{Q_1^+(0,t_0)} (v_0)_t \varphi v_0 dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla(\varphi v_0) dxdt + \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt \\
&= \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt.
\end{aligned}$$

Hence,

$$0 \leq \int_{Q_1^+(0,t_0)} \varphi\,(A_1 - \mathcal{J}(g))\,(\nabla v_0 - \nabla g)\, dxdt$$

holds for all $\varphi \in C_c^1(Q_1^+(0,t_0))$, $\varphi \geq 0$ and $g \in X$. Now we choose $g = v_0 - \xi(v_0 - V_0) = (1-\xi)v_0 + \xi V_0 \in X$ for $\xi \in (0,1)$, so

$$0 \leq \int_{Q_1^+(0,t_0)} \varphi\,(A - \mathcal{J}(v_0 - \xi(v_0 - V_0)))\,(\nabla v_0 - \nabla V_0)\, dxdt$$

Letting $\xi \to 0^+$ and $\varphi \to \chi_{Q_1^+(0,t_0)}$, we get (4.7.56). Similarly, we also obtain (4.7.57). Thus,

$$\int_{Q_1^+(0,t_0)} (A_1 - A_2)\nabla(v_0 - V_0) dxdt \geq \int_{Q_1^+(0,t_0)} (\mathcal{J}(v_0) - \mathcal{J}(V_0))\nabla(v_0 - V_0) dxdt.$$

Combining this with (4.7.54), (4.7.55) and $v_0 - V_0 \in L^2(t_0 - 1, t_0, H_0^1(B_1^+(0)))$, yields $v_0 = V_0$. This completes the proof of Lemma. ■

Lemma 4.7.11 *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0-1, t_0; L^2(\Omega \cap B_1(0))) \cap L^2(t_0-1, t_0; H^1(\Omega \cap B_1(0)))$ is a solution of (4.7.44) and (4.7.45) is satisfied and the bounded*

$$\fint_{Q_1(0,t_0)} |\nabla v|^2 dxdt \leq 1, \tag{4.7.58}$$

then there exists a weak solution $V \in C(t_0-1, t_0; L^2(B_1^+(0))) \cap L^2(t_0-1, t_0; H^1(B_1^+(0)))$ of (4.7.41) with $\rho = 1$, whose zero extension to $Q_1(0, t_0)$ satisfies

$$||\nabla V||_{L^\infty(Q_{1/4}(0,t_0))} \leq C \quad \text{and} \tag{4.7.59}$$

$$\fint_{Q_{1/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \leq \varepsilon^2, \tag{4.7.60}$$

for some $C = C(N, \Lambda_1, \Lambda_2) > 0$.

Proof. Given $\varepsilon_1 \in (0,1)$ by applying Lemma 4.7.10 one finds a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon_1) > 0$ and a weak solution $V \in C(t_0-1, t_0; L^2(B_1^+(0))) \cap L^2(t_0-1, t_0; H^1(B_1^+(0)))$ of (4.7.41) with $\rho = 1$ such that

$$\fint_{Q_1(0,t_0)} |v - V|^2 dxdt \leq \varepsilon_1^2, \tag{4.7.61}$$

Using $\phi^2 V$ with $\phi \in C_c^\infty(B_1 \times (t_0-1, t_0])$, $0 \leq \phi \leq 1$ and $\phi = 1$ in $Q_{1/2}(0,t_0)$ as test function in (4.7.41), we can obtain

$$\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt \leq c_1 \int_{Q_1(0,t_0)} |V|^2 dxdt.$$

This implies

$$\begin{aligned}
\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt &\leq c_2 \int_{Q_1(0,t_0)} \left(|v - V|^2 + |v|^2\right) dxdt \\
&\leq c_3 \int_{Q_1(0,t_0)} \left(|v - V|^2 + |\nabla v|^2\right) dxdt \\
&\leq c_4,
\end{aligned}$$

since (4.7.58), (4.7.61) and Poincare's inequality. Thus, using Lemma 4.7.8 we get (4.7.59). Next, we will prove (4.7.60). By Lemma 4.7.9, the zero extension of V to $Q_1(0, t_0)$ satisfies

$$V_t - div\left(\overline{A}_{B_1(0)}(t, \nabla V)\right) = \frac{\partial F}{\partial x_N} \quad \text{in weakly} \quad Q_1(0, t_0).$$

where $F(x,t) = \chi_{x_N<0} \overline{A}^N_{B_\rho(0)}(t, \nabla V(x', 0, t))$. Thus, we can write

$$\begin{aligned}
&\int_{\tilde{\Omega}_1(0,t_0)} (V - v)_t \varphi dxdt \\
&\quad + \int_{\tilde{\Omega}_1(0,t_0)} \left(\overline{A}_{B_1(0)}(t, \nabla V) - \overline{A}_{B_1(0)}(t, \nabla v)\right) \nabla\varphi dxdt = -\int_{\tilde{\Omega}_1(0,t_0)} F \frac{\partial \varphi}{\partial x_N} dxdt,
\end{aligned}$$

for any $\varphi \in L^2(t_0-1, t_0, H_0^1(\Omega \cap B_1(0)))$.
We take $\varphi = \phi^2(V-v)$ where $\varphi \in C_c^\infty(B_{1/4} \times (t_0-(1/4)^2, t_0])$, $0 \le \phi \le 1$ and $\phi = 1$ on $\overline{Q}_{1/8}(0,t_0)$, so

$$\begin{aligned}
&\int_{\tilde{\Omega}_1(0,t_0)} \phi^2 \left(\overline{A}_{B_1(0)}(t,\nabla V) - \overline{A}_{B_1(0)}(t,\nabla v)\right)(\nabla V - \nabla v)\, dxdt \\
&\quad = -2\int_{\tilde{\Omega}_1(0,t_0)} \phi(V-v)\left(\overline{A}_{B_1(0)}(t,\nabla V) - \overline{A}_{B_1(0)}(t,\nabla v)\right)\nabla\phi dxdt \\
&\quad - \int_{\tilde{\Omega}_1(0,t_0)} \phi^2 (V-v)_t (V-v) dxdt \\
&\quad - \int_{\tilde{\Omega}_1(0,t_0)} \left(\phi^2 F \frac{\partial (V-v)}{\partial x_N} + 2\phi F (V-v)\frac{\partial \phi}{\partial x_N}\right) dxdt.
\end{aligned}$$

We can rewrite $I_1 = I_2 + I_3 + I_4$.
We see that

$$I_1 \ge c_5 \int_{\tilde{\Omega}_1(0,t_0)} \phi^2 |\nabla V - \nabla v|^2 dxdt$$

and using Holder's inequality

$$\begin{aligned}
|I_2| &\le c_6 \int_{\tilde{\Omega}_1(0,t_0)} \phi |V-v|(|\nabla V| + |\nabla v|)|\nabla\phi| dxdt \\
&\le \varepsilon_2 \int_{\tilde{\Omega}_1(0,t_0)} \phi^2(|\nabla V|^2 + |\nabla v|^2) dxdt + c_7(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |V-v|^2 |\nabla\phi|^2 dxdt.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
|I_4| \le\ & \varepsilon_2 \int_{\tilde{\Omega}_1(0,t_0)} \phi^2(|\nabla V|^2 + |\nabla v|^2) dxdt + c_8(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |V-v|^2 |\nabla\phi|^2 dxdt \\
&+ c_8(\varepsilon_2)\int_{\tilde{\Omega}_1(0,t_0)} |F|^2 \phi^2 dxdt,
\end{aligned}$$

and

$$I_3 \le \int_{\tilde{\Omega}_1(0,t_0)} \phi_t \phi (V-v)^2 dxdt \le c_9 \int_{\tilde{\Omega}_{1/4}(0,t_0)} |V-v|^2 dxdt.$$

Hence,

$$\begin{aligned}
&\int_{\tilde{\Omega}_{1/8}(0,t_0)} |\nabla V - \nabla v|^2 \\
&\le c_{10}\varepsilon_2 \int_{\tilde{\Omega}_{1/4}(0,t_0)} (|\nabla V|^2 + |\nabla v|^2) + c_{11}(\varepsilon_2)\int_{\tilde{\Omega}_{1/4}(0,t_0)} (|V-v|^2 + |F|^2) \\
&\le c_{12}\varepsilon_2 + c_{13}(\varepsilon_2)\left(\varepsilon_1^2 + \int_{\tilde{\Omega}_{1/4}(0,t_0)\cap\{-4\delta < x_N < 0\}} |\nabla V(x',0,t)|^2 dxdt\right) \\
&\le\ c_{12}\varepsilon_2 + c_{14}(\varepsilon_2)\left(\varepsilon_1^2 + \delta\right).
\end{aligned}$$

Finally, for any $\varepsilon > 0$ by choosing $\varepsilon_2, \varepsilon_1$ and δ appropriately we get (4.7.60). This completes the proof of Lemma. ∎

Lemma 4.7.12 *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0 - \rho^2, t_0; L^2(\Omega \cap B_\rho(0))) \cap L^2(t_0 - \rho^2, t_0; H^1(\Omega \cap B_\rho(0)))$ is a solution of*

$$\left\{ \begin{array}{ll} v_t - div\left(\overline{A}_{B_\rho(0)}(t, \nabla v)\right) = 0 \; in \; \tilde{\Omega}_\rho(0) \\ v = 0 \qquad on \;\; \partial_p \tilde{\Omega}_\rho(0) \backslash (\Omega \times (-T, T)) \end{array} \right. \tag{4.7.62}$$

and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \tag{4.7.63}$$

then there exists a weak solution $V \in C(t_0 - \rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2(t_0 - \rho^2, t_0; H^1(B_\rho^+(0)))$ of (4.7.41)*, whose zero extension to $Q_1(0, t_0)$ satisfies*

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))} \le C \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt \quad and \tag{4.7.64}$$

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \le \varepsilon^2 \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt. \tag{4.7.65}$$

for some $C = C(N, \Lambda_1, \Lambda_2) > 0$.

Proof. We set

$$\mathcal{A}(x, t, \xi) = A(\rho x, t_0 + \rho^2(t - t_0), \kappa\xi)/\kappa \;\; \text{and} \; \tilde{v}(x, t) = v(\rho x, t_0 + \rho^2(t - t_0))/(\rho\kappa)$$

where $\kappa = \left(\frac{1}{|Q_\rho(0,t_0)|} \int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt\right)^{1/2}$. Then $\mathcal{A}$ satisfies conditions (4.1.2) and (4.2.27) with the same constants Λ_1 and Λ_2. We can see that $\tilde{v}$ is a solution of

$$\left\{ \begin{array}{ll} \tilde{v}_t - div\left(\overline{\mathcal{A}}_{B_1(0)}(t, \nabla \tilde{v})\right) = 0 & \text{in} \;\; \tilde{\Omega}_1^\rho(0) \\ \tilde{v} = 0 & \text{on} \;\; ((\partial\Omega^\rho \cap B_1(0)) \times (t_0 - 1, t_0)) \cup ((\Omega^\rho \cap B_1(0)) \times \{t = t_0 - 1\}) \end{array} \right. \tag{4.7.66}$$

where $\Omega^\rho = \{z = x/\rho : x \in \Omega\}$ and satisfies $\fint_{Q_1(0,t_0)} |\nabla \tilde{v}|^2 dxdt = 1$. We also have

$$B_1^+(0) \subset \Omega^\rho \cap B_1(0) \subset B_1(0) \cap \{x_N > -4\delta\}.$$

Therefore, applying Lemma 4.7.11 for any $\varepsilon > 0$, there exist a constant $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ and $\tilde{V}$ satisfies

$$||\nabla \tilde{V}||_{L^\infty(Q_{1/4}(0,t_0))} \le c_1 \;\; \text{and} \fint_{Q_{1/8}(0,t_0)} |\nabla \tilde{v} - \nabla \tilde{V}|^2 dxdt \le \varepsilon^2.$$

We complete the proof by choosing $V(x, t) = k\rho\tilde{V}(x/\rho, t_0 + (t - t_0)/\rho^2)$. ■

Lemma 4.7.13 *Let s_2 be as in Lemma 4.7.7. For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that the following holds. If Ω is a (δ, R_0)-Reifenberg flat domain and $u \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ is a solution to equation (4.2.4) with $\mu \in L^2(\Omega \times (-T, T))$ and $u(-T) = 0$, for $x_0 \in \partial\Omega$, $-T < t_0 < T$ and $0 < R < R_0/6$ then there is a function $V \in L^2(t_0 - (R/9)^2, t_0; H^1(B_{R/9}(x_0))) \cap L^\infty(t_0 - (R/9)^2, t_0; W^{1,\infty}(B_{R/9}(x_0)))$ such that*

$$||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))} \le c \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + c\frac{|\mu|(Q_{6R}(x_0, t_0))}{R^{N+1}} \tag{4.7.67}$$

and

$$\fint_{Q_{R/9}(x_0,t_0)} |\nabla u - \nabla V| dxdt$$
$$\leq c(\varepsilon + [A]_{s_2}^{R_0}) \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + c(\varepsilon + 1 + [A]_{s_2}^{R_0}) \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}, \qquad (4.7.68)$$

for some $c = c(N, \Lambda_1, \Lambda_2) > 0$.

Proof. Let $x_0 \in \partial\Omega$, $-T < t_0 < T$ and $\rho = R(1-\delta)$, we may assume that $0 \in \Omega$, $x_0 = (0, ..., -\delta\rho/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \qquad (4.7.69)$$

We also have

$$Q_{R/9}(x_0,t_0) \subset Q_{\rho/8}(0,t_0) \subset Q_{\rho/4}(0,t_0) \subset Q_\rho(0,t_0) \subset Q_{6\rho}(0,t_0) \subset Q_{6R}(x_0,t_0), \qquad (4.7.70)$$

provided that $0 < \delta < 1/625$.
Let w and v be in Theorem 4.7.5 and Lemma 4.7.7. By Lemma 4.7.12 for any $\varepsilon > 0$ we can find a small positive $\delta = \delta(N, \alpha, \beta, \varepsilon) < 1/625$ such that there is a function $V \in L^2(t_0 - \rho^2, t_0; H^1(B_\rho(0))) \cap L^\infty(t_0 - \rho^2, t_0; W^{1,\infty}(B_\rho(0)))$ satisfying

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))} \leq c_1 \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt \text{ and}$$
$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|^2 \leq \varepsilon^2 \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt.$$

Then, by (4.7.40) in Lemma 4.7.7 and (4.7.19) in Theorem 4.7.5 and (4.7.70) we get

$$||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))} \leq c_2 \left(\fint_{Q_\rho(0,t_0)} |\nabla w|^2 dxdt \right)^{1/2}$$
$$\leq c_3 \fint_{Q_{6R}(x_0,t_0)} |\nabla w| dxdt \qquad (4.7.71)$$

and

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V| dxdt \leq c_4 \varepsilon \left(\fint_{Q_\rho(0,t_0)} |\nabla w|^2 dxdt \right)^{1/2}$$
$$\leq c_5 \varepsilon \fint_{Q_{6R}(x_0,t_0)} |\nabla w| dxdt. \qquad (4.7.72)$$

Therefore, from (4.7.18) in Theorem 4.7.5 and (4.7.71) we get (4.7.67).
Now we prove (4.7.68), we have

$$\fint_{Q_{R/9}(x_0,t_0)} |\nabla u - \nabla V| dxdt \leq c_6 \fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla V| dxdt$$
$$\leq c_6 \fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla w| dxdt + c_6 \fint_{Q_{\rho/8}(0,t_0)} |\nabla w - \nabla v| dxdt$$
$$+ c_8 \fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V| dxdt.$$

From Lemma 4.7.7 and Theorem 4.7.5 and (4.7.72) it follows that

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla w| dxdt \leq c_7 \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}},$$

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla w| dxdt &\leq c_8 [A]_{s_2}^{R_0} \fint_{Q_{6\rho}(0,t_0)} |\nabla w| dxdt \\ &\leq c_9 [A]_{s_2}^{R_0} \fint_{Q_{6R}(x_0,t_0)} |\nabla w| dxdt \\ &\leq c_{10} [A]_{s_2}^{R_0} \left(\fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}} \right),\end{aligned}$$

and

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V| dxdt &\leq c_{11} \varepsilon \fint_{Q_{6R}(x_0,t_0)} |\nabla w| dxdt \\ &\leq c_{12} \varepsilon \left(\fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}} \right).\end{aligned}$$

Hence we get (4.7.68).

■

4.8 Global Integral Gradient Bounds for Parabolic equations

4.8.1 Global estimates on 2-Capacity uniform thickness domains

We use the Theorem 4.7.1 and 4.7.5 to prove the following theorem.

Theorem 4.8.1 *Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let θ_1, θ_2 be in Theorem 4.7.1 and 4.7.5. Set $\theta = \min\{\theta_1, \theta_2\}$ and $T_0 = diam(\Omega) + T^{1/2}$. Let $Q = B_{diam(\Omega)}(x_0) \times (0,T)$ that contains Ω_T. Let $B_1 = \tilde{Q}_{R_1}(y_0, s_0)$, $B_2 = 4B_1 := \tilde{Q}_{4R_1}(y_0,s_0)$ for $R_1 > 0$. For $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$, there exist a distribution solution u of equation (4.2.4) with data μ, $u_0 = \sigma$ and constants $C_1 = C_1(N, \Lambda_1, \Lambda_2, c_0, T_0/r_0), c_2 > 0$, $\varepsilon_1 = \varepsilon_1(N, \Lambda_1, \Lambda_2, c_0, T_0/r_0), \varepsilon_2 = \varepsilon_1(N, \Lambda_1, \Lambda_2, c_0) > 0$ such that*

$$|\{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \leq \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q| \leq C_1 \varepsilon |\{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q|, \tag{4.8.1}$$

for all $\lambda > 0, \varepsilon \in (0, \varepsilon_1)$ and

$$|\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \leq \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1| \leq C_1 \varepsilon |\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1|, \tag{4.8.2}$$

for all $\lambda > \varepsilon^{-1+\frac{1}{\theta}} ||\nabla u||_{L^1(\Omega_T \cap B_2)} R_2^{-N-2}$, $\varepsilon \in (0, \varepsilon_2)$ with $R_2 = \inf\{r_0, R_1\}/16$. Moreover, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof of Theorem 4.8.1. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \{\sigma_n\} \subset C_c^\infty(\Omega)$ be as in the proof of Theorem 4.2.1. We have $|\mu_n| \le \varphi_n * |\mu|$ and $|\sigma_n| \le \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}, \{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$, respectively.
Let u_n be solution of equation

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{in } \ \Omega. \end{cases} \tag{4.8.3}$$

By Proposition 4.3.5 and Theorem 4.3.6, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ converging to a distribution solution u of (4.2.4) with data $\mu \in \mathfrak{M}_b(\Omega_T)$ and $u_0 = \sigma$ such that $u_n \to u$ in $L^s(0,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.
By Remark 4.3.3 and Theorem 4.3.6, a sequence $\{u_{n,m}\}_m$ of solutions to equations

$$\begin{cases} (u_{n,m})_t - div(A(x,t,\nabla u_{n,m})) = \mu_{n,m} & \text{in } \Omega \times (-T,T), \\ u_{n,m} = 0 \quad \text{on } \partial\Omega \times (-T,T), \\ u_{n,m}(-T) = 0 \ \text{ on } \Omega, \end{cases}$$

converges to $\chi_{\Omega_T} u_n$ in $L^s(-T,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$, where $\mu_{n,m} = (g_{n,m})_t + \chi_{\Omega_T}\mu_n$, $g_{n,m}(x,t) = \sigma_n(x)\int_{-T}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
Set

$$E^1_{\lambda,\varepsilon} = \{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q, \quad F^1_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q,$$
$$E^2_{\lambda,\varepsilon} = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1, \quad F^2_\lambda = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1,$$

for $\varepsilon \in (0,1)$ and $\lambda > 0$.
We verify that

$$|E^1_{\lambda,\varepsilon}| \le c_1\varepsilon|\tilde{Q}_{R_3}| \ \ \forall\ \lambda > 0, \varepsilon \in (0,1) \ \text{ and} \tag{4.8.4}$$

$$|E^2_{\lambda,\varepsilon}| \le c_2\varepsilon|\tilde{Q}_{R_2}| \ \ \forall\ \ \lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap A)}R_2^{-N-2}, \varepsilon \in (0,1) \tag{4.8.5}$$

for some $c_1 = c_1(T_0/r_0), c_2 > 0$ and $R_3 = \inf\{r_0, T_0\}/16$.
In fact, we can assume that $E^1_{\lambda,\varepsilon} \ne \emptyset$ so $(|\mu|(\Omega_T) + |\sigma|(\Omega)) \le T_0^{N+1}\varepsilon^{1-\frac{1}{\theta}}\lambda$. We have

$$|E^1_{\lambda,\varepsilon}| \le \frac{c_3}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}|\nabla u|dxdt.$$

By Remark 4.3.2, $\int_{\Omega_T}|\nabla u_n|dxdt \le c_4T_0\left(|\mu_n|(\Omega_T) + |\sigma_n|(\Omega)\right)$ for all n. Letting $n \to \infty$ we get $\int_{\Omega_T}|\nabla u|dxdt \le c_4T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right)$. Thus,

$$|E^1_{\lambda,\varepsilon}| \le \frac{c_3c_4}{\varepsilon^{-1/\theta}\lambda}T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right) \le \frac{c_3c_4}{\varepsilon^{-1/\theta}\lambda}T_0^{N+2}\varepsilon^{1-\frac{1}{\theta}}\lambda = c_5\varepsilon|\tilde{Q}_{R_3}|.$$

Hence, (4.8.4) holds with $c_1 = c_5(T_0/r_0)$.
For any $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap B_2)}R_2^{-N-2}$ we have

$$|E^2_{\lambda,\varepsilon}| \le \frac{c_3}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}\chi_{B_2}|\nabla u|dxdt < c_2\varepsilon|\tilde{Q}_{R_2}|.$$

Hence, (4.8.5) holds.
Next we verify that for all $(x,t) \in Q$ and $r \in (0, R_3]$ and $\lambda > 0, \varepsilon \in (0,1)$ we have $\tilde{Q}_r(x,t) \cap Q \subset F_\lambda^1$ if $|E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t)| \geq c_6 \varepsilon |\tilde{Q}_r(x,t)|$ where the constant c_6 does not depend on λ and ε. Indeed, take $(x,t) \in Q$ and $0 < r \leq R_3$. Now assume that $\tilde{Q}_r(x,t) \cap Q \cap (F_\lambda^1)^c \neq \emptyset$ and $E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t) \neq \emptyset$ i.e, there exist $(x_1,t_1), (x_2,t_2) \in \tilde{Q}_r(x,t) \cap Q$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \varepsilon^{1-\frac{1}{\theta}}\lambda$. We need to prove that

$$|E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t))| < c_6 \varepsilon |\tilde{Q}_r(x,t)| \tag{4.8.6}$$

Obviously, we have for all $(y,s) \in \tilde{Q}_r(x,t)$ there holds

$$\mathbb{M}(|\nabla u|)(y,s) \leq \max\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s), 3^{N+2}\lambda\}.$$

Leads to, for all $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 \leq 3^{-(N+2)\theta}$,

$$E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t) = \{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \leq \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q \cap \tilde{Q}_r(x,t). \tag{4.8.7}$$

In particular, $E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t) = \emptyset$ if $\overline{B}_{4r}(x) \subset\subset \mathbb{R}^N \backslash \Omega$. Thus, it is enough to consider the case $B_{4r}(x) \subset\subset \Omega$ and $B_{4r}(x) \cap \Omega \neq \emptyset$.
We consider the case $B_{4r}(x) \subset\subset \Omega$. Let $w_{n,m}$ be as in Theorem 4.7.1 with $Q_{2R} = Q_{4r}(x,t_0)$ and $u = u_{n,m}$ where $t_0 = \min\{t + 2r^2, T\}$. We have

$$\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m} - \nabla w_{n,m}| dxdt \leq c_7 \frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}} \quad \text{and} \tag{4.8.8}$$

$$\fint_{Q_{2r}(x,t_0)} |\nabla w_{n,m}|^\theta dxdt \leq c_8 \left(\fint_{Q_{4r}(x,t_0)} |\nabla w_{n,m}| dxdt\right)^\theta. \tag{4.8.9}$$

From (4.8.7), we have

$$\begin{aligned}
|E_{\lambda,\varepsilon}^1 \cap \tilde{Q}_r(x,t)| &\leq |\{\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla w_{n,m}|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)\}| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla w_{n,m}|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla u_n|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\quad + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u|\right) > \varepsilon^{-1/\theta}\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
&\leq c_9 \varepsilon \lambda^{-\theta} \int_{\tilde{Q}_{2r}(x,t)} |\nabla w_{n,m}|^\theta dxdt + c_9 \varepsilon^{1/\theta} \lambda^{-1} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla w_{n,m}| dxdt \\
&\quad + c_9 \varepsilon^{1/\theta} \lambda^{-1} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla u_n| dxdt + c_9 \varepsilon^{1/\theta} \lambda^{-1} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u| dxdt.
\end{aligned}$$

Thanks to (4.8.8) and (4.8.9) we can continue

$$\begin{aligned}|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|&\le c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{4r}(x,t_0)}|\nabla u_{n,m}|dxdt\right)^\theta\\&+c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}\right)^\theta+c_{10}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}\\&+c_{10}\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)}|\nabla u_{n,m}-\nabla u_n|dxdt+c_{10}\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)}|\nabla u_n-\nabla u|dxdt.\end{aligned}$$

Letting $m\to\infty$ and $n\to\infty$, we get

$$\begin{aligned}|E_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|&\le c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{4r}(x,t_0)}|\nabla u|dxdt\right)^\theta\\&+c_{10}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}\right)^\theta+c_{10}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}.\end{aligned}$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1)\le\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\le\varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$\int_{Q_{4r}(x,t_0)}|\nabla u|dxdt\le\int_{\tilde{Q}_{8r}(x,t)}|\nabla u|dxdt\le\int_{\tilde{Q}_{9r}(x_1,t_1)}|\nabla u|dxdt\le|\tilde{Q}_{9r}(x_1,t_1)|\lambda,$$

and

$$\omega(\overline{Q_{4r}(x,t_0)})\le\omega(\tilde{Q}_{8r}(x,t))\le\omega(\tilde{Q}_{9r}(x_2,t_2)\le\varepsilon^{1-\frac{1}{\theta}}\lambda(9r)^{N+1}.$$

Thus

$$|E_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|\le c_{11}\varepsilon|\tilde{Q}_r(x,t)|.$$

Next, we consider the case $B_{4r}(x)\cap\Omega\neq\emptyset$. Let $x_3\in\partial\Omega$ such that $|x_3-x|=\mathrm{dist}(x,\partial\Omega)$. Let w_n be as in Theorem 4.7.5 with $\tilde{\Omega}_{6R}=\tilde{\Omega}_{16r}(x_3,t_0)$ and $u=u_{n,m}$ where $t_0=\min\{t+2r^2,T\}$. We have $Q_{12r}(x,t_0)\subset Q_{16r}(x_3,t_0)$,

$$\fint_{Q_{12r}(x,t_0)}|\nabla u_{n,m}-\nabla w_{n,m}|dxdt\le c_{12}\frac{|\mu_{n,m}|(\tilde{\Omega}_{16r}(x_3,t_0))}{r^{N+1}}\quad\text{and}$$

$$\left(\fint_{Q_{2r}(x,t_0)}|\nabla w_{n,m}|^\theta dxdt\right)^{\frac{1}{\theta}}\le c_{13}\fint_{Q_{12r}(x,t_0)}|\nabla w_{n,m}|dxdt.$$

As above we also obtain

$$\begin{aligned}|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|&\le c_{14}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{12r}(x,t_0)}|\nabla u|dxdt\right)^\theta\\&+c_{14}\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}\right)^\theta+c_{14}\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}.\end{aligned}$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$\int_{Q_{12r}(x,t_0)} |\nabla u|dxdt \leq \int_{\tilde{Q}_{24r}(x,t)} |\nabla u|dxdt \leq \int_{\tilde{Q}_{25r}(x_1,t_1)} |\nabla u|dxdt \leq |\tilde{Q}_{25r}(x_1,t_1)|\lambda$$

and

$$\omega(\overline{Q_{16r}(x_3,t_0)}) \leq \omega(\tilde{Q}_{32r}(x_3,t)) \leq \omega(\tilde{Q}_{36r}(x,t)) \leq \omega(\tilde{Q}_{37r}(x_2,t_2)) \leq \varepsilon^{1-\frac{1}{\theta}}\lambda(37r)^{N+1}.$$

Thus

$$|E^1_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \leq c_{15}\varepsilon|\tilde{Q}_r(x,t)|.$$

Hence, (4.8.6) holds with $c_6 = 2\max\{c_{11}, c_{15}\}$.
Similarly, we also prove that for all $(x,t) \in B_1$ and $r \in (0, R_2]$ and $\lambda > 0, \varepsilon \in (0,1)$ we have $\tilde{Q}_r(x,t) \cap B_1 \subset F^2_\lambda$ if $|E^2_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \geq c_{16}\varepsilon|\tilde{Q}_r(x,t)|$ where a constant c_{26} does not depend on λ and ε. Now, choose $\varepsilon_1 = (2\max\{1, c_1, c_6\})^{-1}$ and $\varepsilon_2 = (2\max\{1, c_2, c_{16}\}^{-1}$. We apply Lemma 4.3.21 with $E = E^1_{\lambda,\varepsilon}, F = F^1_\lambda$ and ε is replaced by $\max\{c_1, c_6\}\varepsilon$ for any $0 < \varepsilon < \varepsilon_1$ and $\lambda > 0$ we get (4.8.1), for $E = E^2_{\lambda,\varepsilon}, F = F^2_\lambda$ and ε is replaced by $\max\{c_1, c_{17}\}\varepsilon$ for any $0 < \varepsilon < \varepsilon_2$ and $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T \cap B_2)}R_2^{-N-2}$ we get (4.8.2).
This completes the proof of the Theorem. ■

Proof of Theorem 4.2.17. By theorem 4.8.1, there exist constants $c_1 > 0$, $0 < \varepsilon_0 < 1$ and a renormalized solution u of equation (4.2.4) with data μ, $u_0 = \sigma$ such that for any $\varepsilon \in (0,1)$, $\lambda > 0$

$$|\{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \leq \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q| \leq c_1\varepsilon|\{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q|.$$

Therefore, if $0 < s < \infty$

$$\begin{aligned}
||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)} &= \varepsilon^{-s/\theta}p\int_0^\infty \lambda^s|\{(x,t) \in Q : \mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda} \\
&\leq c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}}p\int_0^\infty \lambda^s|\{(x,t) \in Q : \mathbb{M}(|\nabla u|) > \lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda} \\
&+ \varepsilon^{-s/\theta}p\int_0^\infty \lambda^s|\{(x,t) \in Q : \mathbb{M}_1[\omega] > \varepsilon^{1-\frac{1}{\theta}}\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda} \\
&= c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}}||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)} + \varepsilon^{-s}||\mathbb{M}_1[\omega]||^s_{L^{p,s}(Q)}.
\end{aligned}$$

Since $p < \theta$, we can choose $0 < \varepsilon < \varepsilon_0$ such that $c_1^{s/p}\varepsilon^{\frac{s(\theta-p)}{\theta p}} \leq 1/2$ we get the result for case $0 < s < \infty$. Similarly, we also get the result for case $s = \infty$.
Also, we get (4.2.29) by using (4.4.16) in Proposition 4.4.8, (4.4.28) in Proposition 4.4.19.
This completes the proof. ■

Remark 4.8.2 *Thanks to Proposition 4.4.4 we have for any* $s \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ *if* $\mu \in L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)$ *and* $\sigma \equiv 0$ *then*

$$|||\nabla u|^s||_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)} \leq c_2||\mu||^s_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)},$$

where constant c_2 *depends on* $N, \Lambda_1, \Lambda_2, s, c_0, T_0/r_0$.

As the proof of Theorem 4.8.1, we also get

Theorem 4.8.3 *Suppose that $\mathbb{R}^N\backslash\Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let θ be as in Theorem 4.8.1. Let $1 \leq p < \theta$, $0 < s \leq \infty$ and $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exist $C_1 = C_1(N, \Lambda_1, \Lambda_2, p, s, c_0) > 0$ and a distribution solution u of equation (4.2.4) with data μ and $u_0 = \sigma$ such that*

$$\begin{aligned}||\mathbb{M}(\chi_{\tilde{Q}_{4R}(y_0,s_0)}|\nabla u|)||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} &\leq C_1 R^{\frac{N+2}{p}} \inf\{r_0, R\}^{-N-2}||\nabla u||_{L^1(\tilde{Q}_{4R}(y_0,s_0))} \\ &+ C_1||\mathbb{M}_1[\chi_{\tilde{Q}_{4R}(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}, \qquad (4.8.10)\end{aligned}$$

for any $\tilde{Q}_R(y_0, s_0) \subset \mathbb{R}^{N+1}$ and if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof of Theorem 4.2.19. Let $\{u_{n,m}\}$ and $\mu_{n,m}$ be in the proof of Theorem 4.8.1. From Corollary 4.7.2 and 4.7.6 we assert : for $2-\inf\{\beta_1, \beta_2\} < \gamma < N+2$, there exists a constant $C = C(N, \Lambda_1, \Lambda_2, c_0, \gamma) > 0$ such that for any $0 < \rho \leq T_0$

$$\int_{Q_\rho(y,s)} |\nabla u_{n,m}| dxdt \leq C(N, \Lambda_1, \Lambda_2, \gamma, c_0, T_0/r_0)\rho^{N+3-\gamma}||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))},$$

where β_1, β_2 are constants in Theorem 4.7.1 and Theorem 4.7.5. It is easy to see that

$$||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))} \leq ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega\times(-T,T))} = ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)},$$

for any n, m large enough.
Letting $m \to \infty, n \to \infty$, yield

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \leq C(N, \Lambda_1, \Lambda_2, \gamma, c_0, T_0/r_0)\rho^{N+3-\gamma}||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)}$$

By Theorem 4.8.3 we get

$$\begin{aligned}|||\nabla u|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0)\cap\Omega_T)} &\leq c_1(T_0/r_0)R^{\frac{N+2}{p}+1-\gamma}||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)} \\ &+ c_2||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\end{aligned}$$

for any $\tilde{Q}_R(y_0, s_0) \subset \mathbb{R}^{N+1}$ and $0 < R \leq T_0$. It follows (4.2.30).

Finally, if $\mu \in L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$ and $\sigma \equiv 0$, then clearly u is a unique renormalized solution. It suffices to show that

$$||\mathbb{M}_\gamma[|\mu|]||_{L^\infty(\Omega_T)} \leq c_3||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} \quad \text{and} \qquad (4.8.11)$$

$$R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \leq c_3||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma}, \frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} \qquad (4.8.12)$$

for any $\tilde{Q}_R(y_0,s_0) \subset \mathbb{R}^{N+1}$ and $0 < R \le T_0$, where $c_3 = c_3(N,\Lambda_1,\Lambda_2,p,s,\gamma,c_0,T_0/r_0)$. In fact, for $0<\rho<T_0$ and $(x,t) \in \Omega_T$ we have

$$\begin{aligned} ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} &\ge ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\infty;(\gamma-1)p}(\Omega_T)} \\ &\ge \rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}} ||\mu||_{L^{\frac{(\gamma-1)p}{\gamma},\infty}(\tilde{Q}_\rho(x,t)\cap\Omega_T)} \\ &\ge c_4\rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}} |\tilde{Q}_\rho(x,t)|^{-1+\frac{\gamma}{(\gamma-1)p}} |\mu|(\tilde{Q}_\rho(x,t)\cap\Omega_T) \\ &= c_5\frac{|\mu|(\tilde{Q}_\rho(x,t)\cap\Omega_T)}{\rho^{N+2-\gamma}}, \end{aligned}$$

which obviously implies (4.8.11).
Next, we note that

$$\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|](x,t) \le c_6\left(\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)(x,t)\right)^{1-\frac{1}{\gamma}} ||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}.$$

We derive

$$\begin{aligned} &R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \\ &\le c_6R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))} ||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} \\ &\le c_7R^{\frac{p(\gamma-1)-N-2}{p}} |||\mu|||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))} ||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}. \end{aligned}$$

Here we used the boundedness property of $\mathbb{M}$ in $L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\mathbb{R}^{N+1})$ for $\frac{(\gamma-1)p}{\gamma} > 1$. Therefore, immediately we get (4.8.12). This completes the proof of theorem.

■

4.8.2 Global estimates on Reifenberg flat domains

Now we prove results for Reifenberg flat domain. First, we will use Lemma 4.7.4, 4.7.13 and Lemma 4.3.19 to get the following result.

Theorem 4.8.4 *Suppose that A satisfies* (4.2.27). *Let s_1,s_2 be in Lemma 4.7.3 and 4.7.7, set $s_0=\max\{s_1,s_2\}$. Let $w \in A_\infty$, $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distribution solution of* (4.2.4) *with data μ and $u_0=\sigma$ such that following holds. For any $\varepsilon > 0, R_0 > 0$ one finds $\delta_1 = \delta_1(N,\Lambda_1,\Lambda_2,\varepsilon,[w]_{A_\infty}) \in (0,1)$ and $\delta_2 = \delta_2(N,\Lambda_1,\Lambda_2,\varepsilon,[w]_{A_\infty},T_0/R_0) \in (0,1)$ and $\Lambda = \Lambda(N,\Lambda_1,\Lambda_2) > 0$ such that if Ω is (δ_1,R_0)-Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \le \delta_1$ then*

$$w(\{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \le \delta_2\lambda\} \cap \Omega_T) \le B\varepsilon w(\{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T) \qquad (4.8.13)$$

for all $\lambda > 0$, where the constant B depends only on $N,\Lambda_1,\Lambda_2,T_0/R_0,[w]_{A_\infty}$. Furthermore, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof. Let $\{\mu_n\}, \{\sigma_n\}, \{\mu_{n,m}\}, \{u_n\}, \{u_{n,m}\}, u$ be as in the proof of Theorem 4.8.1. Let ε be in $(0,1)$. Set $E_{\lambda,\delta_2} = \{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \leq \delta_2\lambda\} \cap \Omega_T$ and $F_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T$ for $\varepsilon \in (0,1)$ and $\lambda > 0$. Let $\{y_i\}_{i=1}^L \subset \Omega$ and a ball B_0 with radius $2T_0$ such that

$$\Omega \subset \bigcup_{i=1}^{L} B_{r_0}(y_i) \subset B_0$$

where $r_0 = \min\{R_0/1080, T_0\}$. Let $s_j = T - jr_0^2/2$ for all $j = 0, 1, ..., [\frac{2T}{r_0^2}]$ and $Q_{2T_0} = B_0 \times (T - 4T_0^2, T)$. So,

$$\Omega_T \subset \bigcup_{i,j} Q_{r_0}(y_i, s_j) \subset Q_{2T_0}.$$

We verify that

$$w(E_{\lambda,\delta_2}) \leq \varepsilon w(\tilde{Q}_{r_0}(y_i, s_j)) \;\; \forall\ \lambda > 0 \tag{4.8.14}$$

for some δ_2 small enough, depended on $n, p, \alpha, \beta, \epsilon, [w]_{A_\infty}, T_0/R_0$.
In fact, we can assume that $E_{\lambda,\delta_2} \neq \emptyset$ so $|\mu|(\Omega_T) + |\sigma|(\Omega) \leq T_0^{N+1}\delta_2\lambda$. We have

$$|E_{\lambda,\delta_2}| \leq \frac{c_1}{\Lambda\lambda} \int_{\Omega_T} |\nabla u| dxdt.$$

We also have

$$\int_{\Omega_T} |\nabla u| dxdt \leq c_2 T_0(|\mu|(\Omega_T) + |\sigma|(\Omega)).$$

Thus,

$$|E_{\lambda,\varepsilon}| \leq \frac{c_3}{\Lambda\lambda} T_0(|\mu|(\Omega_T) + |\sigma|(\Omega)) \leq \frac{c_3}{\Lambda\lambda} T_0^{N+2}\delta_2\lambda = c_4\delta_2|Q_{2T_0}|.$$

which implies

$$w(E_{\lambda,\delta_2}) \leq A \left(\frac{|E_{\lambda,\delta_2}|}{|Q_{2T_0}|}\right)^\nu w(Q_{2T_0}) \leq A\left(c_4\delta_2\right)^\nu w(Q_{2T_0})$$

where (A, ν) is a pair of A_∞ constants of w. It is known that (see, e.g [33]) there exist $A_1 = A_1(N, A, \nu)$ and $\nu_1 = \nu_1(N, A, \nu)$ such that

$$\frac{w(\tilde{Q}_{2T_0})}{w(\tilde{Q}_{r_0}(y_i, s_j))} \leq A_1 \left(\frac{|\tilde{Q}_{2T_0}|}{|\tilde{Q}_{r_0}(y_i, s_j)|}\right)^{\nu_1} \quad \forall i, j.$$

So,

$$w(E_{\lambda,\delta_2}) \leq A\left(c_4\delta_2\right)^\nu A_1 \left(\frac{|\tilde{Q}_{T_0}|}{|\tilde{Q}_{r_0}(y_i, s_j)|}\right)^{\nu_1} w(\tilde{Q}_{r_0}(y_i, s_j)) < \varepsilon w(\tilde{Q}_{r_0}(y_i, s_j)) \;\; \forall\ i, j$$

where $\delta_2 \leq \left(\frac{\varepsilon}{2c_5(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$. It follows (4.8.14).
Next we verify that for all $(x,t) \in \Omega_T$ and $r \in (0, 2r_0]$ and $\lambda > 0$ we have $\tilde{Q}_r(x,t) \cap \Omega_T \subset F_\lambda$ if $w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) \geq \varepsilon w(Q_r(x,t))$ for some $\delta_2 \leq \left(\frac{\varepsilon}{2c_5(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$.

Indeed, take $(x,t) \in \Omega_T$ and $0 < r \leq 2r_0$. Now assume that $\tilde{Q}_r(x,t) \cap \Omega_T \cap F_\lambda^c \neq \emptyset$ and $E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) \neq \emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2) \in \tilde{Q}_r(x,t) \cap \Omega_T$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \delta_2\lambda$. We need to prove that

$$w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t))) < \varepsilon w(\tilde{Q}_r(x,t)). \tag{4.8.15}$$

Clearly,

$$\mathbb{M}(|\nabla u|)(y,s) \leq \max\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s), 3^{N+2}\lambda\} \quad \forall (y,s) \in \tilde{Q}_r(x,t).$$

Therefore, for all $\lambda > 0$ and $\Lambda \geq 3^{N+2}$,

$$E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) = \{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right) > \Lambda\lambda, \mathbb{M}_1[\omega] \leq \delta_2\lambda\} \cap \Omega_T \cap \tilde{Q}_r(x,t). \tag{4.8.16}$$

In particular, $E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) = \emptyset$ if $\overline{B}_{8r}(x) \subset\subset \mathbb{R}^N \backslash \Omega$. Thus, it is enough to consider the case $B_{8r}(x) \subset\subset \Omega$ and $B_{8r}(x) \cap \Omega \neq \emptyset$.
We consider the case $B_{8r}(x) \subset\subset \Omega$. Let $v_{n,m}$ be as in Lemma 4.7.4 with $Q_{2R} = Q_{8r}(x,t_0)$ and $u = u_{n,m}$ where $t_0 = \min\{t + 2r^2, T\}$. We have

$$||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \leq c_6 \fint_{Q_{8r}(x,t_0)} |\nabla u_{n,m}| dxdt + c_6 \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}}, \tag{4.8.17}$$

and

$$\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m} - \nabla v_{n,m}| dxdt \leq c_8 \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}} + c_8 [A]_{s_0}^{R_0} \left(\fint_{Q_{8r}(x,t_0)} |\nabla u_{n,m}| dxdt \right.$$
$$\left. + \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}} \right).$$

Thanks to $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \delta_2\lambda$ with $(x_1,t_1),(x_2,t_2) \in Q_r(x,t)$, we get

$$\begin{aligned}
\limsup_{n\to\infty} \limsup_{m\to\infty} ||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t))} &\leq c_9 \fint_{\tilde{Q}_{17r}(x_1,t_1)} |\nabla u| dxdt + c_9 \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}} \\
&\leq c_9\lambda + c_9\delta_2\lambda \\
&\leq c_{10}\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} \limsup_{m\to\infty} \fint_{Q_{4r}(x,t_0)} |\nabla u_n - \nabla v_n| dxdt \\
&\quad \leq c_{11} \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}} + c_{11}[A]_{s_0}^{R_0} \left(\fint_{\tilde{Q}_{17r}(x_1,t_1)} |\nabla u| dxdt + \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}} \right) \\
&\quad \leq c_{11}\delta_2\lambda + c_{11}[A]_{s_0}^{R_0} (\lambda + \delta_2\lambda) \\
&\quad \leq c_{11} (\delta_2 + \delta_1(1+\delta_2)) \lambda.
\end{aligned}$$

Here we used $[A]_{s_0}^{R_0} \leq \delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla v_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))} = ||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \leq 2c_{10}\lambda \text{ and} \tag{4.8.18}$$

$$\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m} - \nabla v_{n,m}| dxdt \leq 2c_{11} \left(\delta_2 + \delta_1(1+\delta_2)\right) \lambda, \tag{4.8.19}$$

for all $n \geq n_0$ and $m \geq k_n$.
In view of (4.8.18) we see that for $\Lambda \geq \max\{3^{N+2}, 8c_{10}\}$ and $n \geq n_0$, $m \geq k_n$,

$$|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla v_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| = 0.$$

Leads to

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \leq & |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla v_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
& + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\
& + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u - \nabla u_n|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)|.
\end{aligned}$$

Therefore, by (4.8.19) and $\tilde{Q}_{2r}(x,t) \subset Q_{4r}(x,t_0)$ we obtain for any $n \geq n_0$ and $m \geq k_n$

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \leq & \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla v_{n,m}| dxdt \\
& + \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| dxdt + \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n| dxdt \\
& \leq c_{13} \left(\delta_2 + \delta_1(1+\delta_2)\right) |Q_r(x,t)| \\
& + \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| dxdt + \frac{c_{12}}{\lambda} \int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n| dxdt.
\end{aligned}$$

Letting $m \to \infty$ and $n \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \leq c_{13} \left(\delta_2 + \delta_1(1+\delta_2)\right) |\tilde{Q}_r(x,t)|.$$

Thus,

$$\begin{aligned}
w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) & \leq C \left(\frac{|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^\nu w(\tilde{Q}_r(x,t)) \\
& \leq C \left(c_{13} \left(\delta_2 + \delta_1(1+\delta_2)\right)\right)^\nu w(\tilde{Q}_r(x,t)) \\
& < \varepsilon w(\tilde{Q}_r(x,t)).
\end{aligned}$$

where δ_2, δ_1 are appropriately chosen, (C,ν) is a pair of A_∞ constants of w.
Next we consider the case $B_{8r}(x) \cap \Omega \neq \emptyset$. Let $x_3 \in \partial\Omega$ such that $|x_3 - x| = \text{dist}(x, \partial\Omega)$.
Set $t_0 = \min\{t + 2r^2, T\}$. We have

$$Q_{2r}(x,t_0) \subset Q_{10r}(x_3,t_0) \subset Q_{540r}(x_3,t_0) \subset \tilde{Q}_{1080r}(x_3,t) \subset \tilde{Q}_{1088r}(x,t) \subset \tilde{Q}_{1089r}(x_1,t_1) \tag{4.8.20}$$

and

$$Q_{540r}(x_3,t_0)\subset\tilde{Q}_{1080r}(x_3,t)\subset\tilde{Q}_{1088r}(x,t)\subset\tilde{Q}_{1089r}(x_2,t_2) \tag{4.8.21}$$

Let $V_{n,m}$ be as in Lemma 4.7.13 with $Q_{6R}=Q_{540r}(x_3,t_0)$, $u=u_{n,m}$ and $\varepsilon=\delta_3\in(0,1)$. We have

$$||\nabla V_{n,m}||_{L^\infty(Q_{10r}(x_3,t_0))}\leq c_{14}\fint_{Q_{540r}(x_3,t_0)}|\nabla u_{n,m}|dxdt+c_{14}\frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}$$

and

$$\begin{aligned}&\fint_{Q_{10r}(x_3,t_0)}|\nabla u_{n,m}-\nabla V_{n,m}|dxdt\\&\quad\leq c_{15}(\delta_3+[A]_{s_0}^{R_0})\fint_{Q_{540r}(x_3,t_0)}|\nabla u_{n,m}|dxdt+c_{15}(\delta_3+1+[A]_{s_0}^{R_0})\frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}.\end{aligned}$$

Since $\mathbb{M}(|\nabla u|)(x_1,t_1)\leq\lambda$, $\mathbb{M}_1[\omega](x_2,t_2)\leq\delta_2\lambda$ and (4.8.20), (4.8.21) we get

$$\begin{aligned}\limsup_{n\to\infty}\limsup_{m\to\infty}||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))}&\leq\limsup_{n\to\infty}\limsup_{m\to\infty}||\nabla V_{n,m}||_{L^\infty(Q_{10r}(x_3,t_0))}\\&\leq c_{14}\fint_{Q_{540r}(x_3,t_0)}|\nabla u|dxdt+c_{14}\frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{R^{N+1}}\\&\leq c_{15}\fint_{\tilde{Q}_{1089r}(x_1,t_1)}|\nabla u|dxdt+c_{15}\frac{\omega(\tilde{Q}_{1089r}(x_2,t_2))}{R^{N+1}}\\&\leq c_{16}\lambda+c_{16}\delta_2\lambda\\&\leq c_{17}\lambda\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{m\to\infty}\fint_{Q_{2r}(x,t_0)}|\nabla u_{n,m}-\nabla V_{n,m}|dxdt\\&\quad\leq c_{18}(\delta_3+[A]_{s_0}^{R_0})\fint_{Q_{540r}(x_3,t_0)}|\nabla u|dxdt+c_{18}(\delta_3+1+[A]_{s_0}^{R_0})\frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{r^{N+1}}\\&\quad\leq c_{19}(\delta_3+[A]_{s_0}^{R_0})\fint_{\tilde{Q}_{1089r}(x_1,t_1)}|\nabla u|dxdt+c_{19}(\delta_3+1+[A]_{s_0}^{R_0})\frac{\omega(\tilde{Q}_{1089}(x_2,t_2))}{r^{N+1}}\\&\quad\leq c_{20}(\delta_3+[A]_{s_0}^{R_0})\lambda+c_{21}(\delta_3+1+[A]_{s_0}^{R_0})\delta_2\lambda\\&\quad\leq c_{20}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)\lambda.\end{aligned}$$

Here we used $[A]_s^{R_0}\leq\delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla V_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))}=||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))}\leq 2c_{17}\lambda\quad\text{and} \tag{4.8.22}$$

$$\fint_{Q_{2r}(x,t_0)}|\nabla u_{n,m}-\nabla V_{n,m}|dxdt\leq 2c_{21}\left((\delta_3+\delta_1)+(\delta_3+1+\delta_1)\delta_2\right)\lambda, \tag{4.8.23}$$

for all $n \geq n_0$ and $m \geq k_n$.
Now set $\Lambda = \max\{3^{N+2}, 8c_{10}, 8c_{17}\}$. As above we also have for $n \geq n_0$, $m \geq k_n$

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| &\leq |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla V_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ &+ |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ &+ |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u - \nabla u_n|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)|.\end{aligned}$$

Therefore from (4.8.23) we obtain

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| &\leq \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla V_{n,m}|dxdt \\ &+ \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}|dxdt + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n|dxdt \\ &\leq c_{23}\left((\delta_3+\delta_1) + (\delta_3+1+\delta_1)\delta_2\right)|\tilde{Q}_r(x,t)| \\ &+ \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}|dxdt + \frac{c_{22}}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n|dxdt.\end{aligned}$$

Letting $m \to \infty$ and $n \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \leq c_{22}\left((\delta_3+\delta_1) + (\delta_3+1+\delta_1)\delta_2\right)|\tilde{Q}_r(x,t)|.$$

Thus

$$\begin{aligned}w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) &\leq C\left(\frac{|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^{\nu} w(\tilde{Q}_r(x,t)) \\ &\leq C\left(c_{22}\left((\delta_3+\delta_1) + (\delta_3+1+\delta_1)\delta_2\right)\right)^{\nu} w(\tilde{Q}_r(x,t)) \\ &< \varepsilon w(\tilde{Q}_r(x,t)),\end{aligned}$$

where $\delta_3, \delta_1, \delta_2$ are appropriately chosen, (C, ν) is a pair of A_∞ constants of w.
Therefore, for all $(x,t) \in \Omega_T$ and $r \in (0, 2r_0]$ and $\lambda > 0$ if $w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) \geq \varepsilon w(\tilde{Q}_r(x,t))$ then $\tilde{Q}_r(x,t) \cap \Omega_T \subset F_\lambda$ where $\delta_1 = \delta_1(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}) \in (0,1)$ and $\delta_2 = \delta_2(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}, T_0/R_0) \in (0,1)$. Applying Lemma 4.3.19 we get the result. ■

Proof of Theorem 4.2.20. As in the proof of Theorem 4.2.17, we can prove (4.2.32) by using estimate (4.8.13) in Theorem 4.8.4. In particular, thanks to Proposition 4.4.4 for $q > \frac{N+2}{N+1}$, $\mu \in L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)$ and $\sigma \equiv 0$,

$$|||\nabla u|^q||_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)} \leq c||\mu||^q_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)}, \tag{4.8.24}$$

where the constant c depends only on $N, \Lambda_1, \Lambda_2, q$ and T_0/R_0. ■

Proof of Theorem 4.2.22. By Theorem 4.2.20, there exists a renormalized solution of (4.2.4) with data μ, $u(0) = \sigma$ satisfied

$$\int_{\Omega_T} |\nabla u|^q dw \leq c_1 \int_{\Omega_T} \left(\mathbb{M}_1[\omega]\right)^q dw \tag{4.8.25}$$

for any $w \in A_\infty$, where $c_1 = c_1(N, \Lambda_1, \Lambda_2, q, T_0/R_0, [w]_{A_\infty})$.
For $0 < \delta < 1$ we have $\mathbb{M}_1[\omega] \leq c_2 \mathbb{I}_1^{2T_0,\delta}[\omega]$ in Ω_T. Thus, (4.8.25) can be rewritten

$$\int_{\Omega_T} |\nabla u|^q dw \leq c_1 c_2^q \int_{\Omega_T} \left(\mathbb{I}_1^{2T_0,\delta}[\omega]\right)^q dw. \tag{4.8.26}$$

Thanks to Proposition 4.4.23 and Corollary 4.4.39 and 4.4.38 we obtain the result. ■

In follow that we usually employ the the Minkowski inequality, for convenience we recall it, for any $0 < q_1 \leq q_2 < \infty$ there holds

$$\left(\int_X \left(\int_Y |f(x,y)|^{q_1} d\mu_2(y)\right)^{\frac{q_2}{q_1}} d\mu_1(x)\right)^{\frac{1}{q_2}} \leq \left(\int_Y \left(\int_X |f(x,y)|^{q_2} d\mu_1(x)\right)^{\frac{q_1}{q_2}} d\mu_2(y)\right)^{\frac{1}{q_1}}$$

for any measure function f in $X \times Y$, where μ_1, μ_2 are nonnegative measure in X and Y respectively.
Proof of Theorem 4.2.21. We will consider only the case $s \neq \infty$ and leave the case $s = \infty$ to the readers. Take $\kappa_1 \in (0, \kappa)$. It is easy to see that for $(x_0, t_0) \in \Omega_T$ and $0 < \rho < \mathrm{diam}(\Omega) + T^{1/2}$

$$w(x,t) = \min\{\rho^{-N-2+\kappa-\kappa_1}, \max\{|x - x_0|, \sqrt{2|t - t_0|}\}^{-N-2+\kappa-\kappa_1}\} \in A_\infty$$

where $[w]_{A_\infty}$ is independent of (x_0, t_0) and ρ. Thus, from (4.2.32) in Theorem 4.2.20 we have

$$\begin{aligned}
||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T)} &= \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T, dw)} \\
&\leq c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\Omega_T, dw)} \\
&= q c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} \int_0^\infty (\lambda^q w(\{\mathbb{M}_1[\omega] > \lambda\} \cap \Omega_T))^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&= q c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} \int_0^\infty \left(\lambda^q \int_0^\infty |\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&=: c_1 \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} A.
\end{aligned} \tag{4.8.27}$$

Since $w \leq \rho^{-N-2+\kappa-\kappa_1}$ and $\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \subset \{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0)$,

$$A \leq q \int_0^\infty \left(\lambda^q \int_0^{\rho^{-N-2+\kappa-\kappa_1}} |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}.$$

We divide to two cases.
Case 1 : $0 < s \leq q$. We can verify that for any nonincreasing function F in $(0, \infty)$ and $0 < a \leq 1$ we have

$$\left(\int_0^\infty F(\tau) d\tau\right)^a \leq 4 \int_0^\infty (\tau F(\tau))^a \frac{d\tau}{\tau}.$$

Hence,

$$\begin{aligned}
A &\le 4q\int_0^\infty\int_0^{\rho^{-N-2+\kappa-\kappa_1}}\left(\lambda^q\tau|\{\mathbb{M}_1[\omega]>\lambda\}\cap\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T|\right)^{\frac{s}{q}}\frac{d\tau}{\tau}\frac{d\lambda}{\lambda}\\
&= 4q\int_0^{\rho^{-N-2+\kappa-\kappa_1}}\int_0^\infty\left(\lambda^q|\{\mathbb{M}_1[\omega]>\lambda\}\cap\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T|\right)^{\frac{s}{q}}\frac{d\lambda}{\lambda}\tau^{\frac{s}{q}}\frac{d\tau}{\tau}\\
&= 4\int_0^{\rho^{-N-2+\kappa-\kappa_1}}||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T)}\tau^{\frac{s}{q}}\frac{d\tau}{\tau}\\
&\le 4\int_0^{\rho^{-N-2+\kappa-\kappa_1}}||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}}\tau^{\frac{s}{q}}\frac{d\tau}{\tau}\\
&= c_2||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Case 2 : $s>q$. Using the Minkowski inequality, yields

$$\begin{aligned}
A &\le c_3\left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}}\left(\int_0^\infty\left(\lambda^q|\{\mathbb{M}_1[\omega]>\lambda\}\cap\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T|\right)^{\frac{s}{q}}\frac{d\lambda}{\lambda}\right)^{\frac{q}{s}}d\tau\right)^{\frac{s}{q}}\\
&\le c_4\left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}}\left(||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}}\right)^{\frac{q}{s}}d\tau\right)^{\frac{s}{q}}\\
&= c_5||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Therefore, we always have

$$A\le c_6||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\rho^{-\frac{s\kappa_1}{q}}.$$

which implies (4.2.33) from (4.8.27).

Similarly, we obtain estimate (4.2.46) by adapting

$$w(x,t)=\min\{\rho^{-N+\vartheta-\vartheta_1},|x-x_0|^{-N+\vartheta-\vartheta_1}\}\in A_\infty$$

in above argument, where $0<\vartheta_1<\vartheta$, $x_0\in\Omega$ and $0<\rho<\mathrm{diam}(\Omega)$ and $[w]_{A_\infty}$ is independent of x_0 and ρ.

Next, to archive (4.2.35) we need to show that for any ball $B_\rho\subset\mathbb{R}^N$

$$\left(\int_0^T|\mathrm{osc}_{B_\rho\cap\overline{\Omega}}u(t)|^qdt\right)^{\frac{1}{q}}\le c_7\rho^{1-\frac{\vartheta}{q}}|||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)} \tag{4.8.28}$$

Since the extension of u over $(\Omega_T)^c$ is zero and $u\in L^1(0,T,W^{1,1}_0(\Omega))$ thus we have for a.e $t\in(0,T)$, $u(.,t)\in W^{1,1}(\mathbb{R}^N)$. Applying [32, Lemma 7.16] to a ball $B_\rho\subset\mathbb{R}^N$, we get for a.e $t\in(0,T)$ and $x\in B_\rho$

$$\begin{aligned}
|u(x,t)-u_{B\rho}(t)| &\le \frac{2^N}{N|B_1(0)|}\int_{B_\rho}\frac{|\nabla u(y,t)|}{|x-y|^{N-1}}dy\\
&\le \frac{2^N}{N|B_1(0)|}\int_{B_{2\rho}(x)}\frac{|\nabla u(y,t)|}{|x-y|^{N-1}}dy\\
&\le c_8\int_0^{3\rho}\frac{\int_{B_r(x)}|\nabla u(y,t)|dy}{r^{N-1}}\frac{dr}{r},
\end{aligned}$$

here $u_{B\rho}(t)$ is the average of $u(.,t)$ over B_ρ, i.e $u_{B\rho}(t) = \frac{1}{|B_\rho|}\int_{B_\rho} u(x,t)dx$.
Using the Minkowski and the Holder inequality, we discover that for a.e $x \in B_\rho$

$$\begin{aligned}
\left(\int_0^T |u(x,t) - u_{B\rho}(t)|^q dt\right)^{\frac{1}{q}} &\le c_8 \left(\int_0^T \left(\int_0^{3\rho} \frac{\int_{B_r(x)} |\nabla u(y,t)|dy}{r^{N-1}} \frac{dr}{r}\right)^q dt\right)^{\frac{1}{q}} \\
&\le c_8 \int_0^{3\rho} \int_{B_r(x)} \left(\int_0^T |\nabla u(y,t)|^q dt\right)^{\frac{1}{q}} dy \frac{dr}{r^N} \\
&\le c_8 \int_0^{3\rho} \left(\int_{B_r(x)} \int_0^T |\nabla u(y,t)|^q dtdy\right)^{\frac{1}{q}} |B_r(x)|^{\frac{q-1}{q}} \frac{dr}{r^N} \\
&\le c_8 |B_1(x)|^{\frac{q-1}{q}} \int_0^{3\rho} r^{\frac{N-\vartheta}{q}} r^{\frac{N(q-1)}{q}} \frac{dr}{r^N} |||\nabla u|||_{L_{**}^{q;\vartheta}(\Omega_T)} \\
&= c_9 \rho^{1-\frac{\vartheta}{q}} |||\nabla u|||_{L_{**}^{q;\vartheta}(\Omega_T)}.
\end{aligned}$$

Therefore, we find (4.8.28) with $c_7 = 2c_9$. ∎

Proof of Proposition 4.2.28. Clearly, estimate (4.2.46) is followed by (4.4.12) in Proposition 4.4.7. We want to emphasize that almost every estimates in this proof will be used the Minkowski inequality. For a ball $B_\rho \subset \mathbb{R}^N$, we have for a.e $x \in \mathbb{R}^N$

$$\begin{aligned}
||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} &= \left(\int_{-\infty}^{+\infty} \left(\int_0^\infty \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+1}} \frac{dr}{r}\right)^q dt\right)^{\frac{1}{q}} \\
&\le \int_0^\infty \left(\int_{-\infty}^{+\infty} (\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \frac{dr}{r^{N+2}}. \qquad (4.8.29)
\end{aligned}$$

Now, we need to estimate $\left(\int_{-\infty}^{+\infty} (\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}}$.
b. We have

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} (\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} &= \left(\int_{-\infty}^{+\infty} \left(\int_{\mathbb{R}^{N+1}} \chi_{\tilde{Q}_r(x,t)}(x_1,t_1) d\mu(x_1,t_1)\right)^q dt\right)^{\frac{1}{q}} \\
&\le \int_{\mathbb{R}^{N+1}} \left(\int_{-\infty}^{+\infty} \chi_{\tilde{Q}_r(x,t)}(x_1,t_1) dt\right)^{\frac{1}{q}} d\mu(x_1,t_1) \\
&= r^{\frac{2}{q}} \mu_1(B_r(x))
\end{aligned}$$

Combining this with (4.8.29) we obtain (4.2.47) and (4.2.49).
Thus, we also assert (4.2.49) from [1, Theorem 3.1].
c. Set $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})} dx$. Using Holder's inequality, yields

$$\mu(\tilde{Q}_r(x,t)) \le r^{\frac{2(q_1-1)}{q_1}} \int_{B_r(x)} \left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}} (w(x_1,t_1))^{q_1} dt_1\right)^{\frac{1}{q_1}} dx_1.$$

Leads to

$$\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \le r^{\frac{2(q_1-1)}{q_1}}\int_{B_r(x)}\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{1}{q}}dx_1.$$

Note that

$$\begin{aligned}
&\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{q_1}{q}}\\
&\qquad= \left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\chi_{\left(t-\frac{\rho^2}{2},t+\frac{\rho^2}{2}\right)}(t_1)(w(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}}dt\right)^{\frac{q_1}{q}}\\
&\qquad\le \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\chi_{\left(t-\frac{\rho^2}{2},t+\frac{\rho^2}{2}\right)}(t_1)dt\right)^{\frac{q_1}{q}}(w(x_1,t_1))^{q_1}dt_1\\
&\qquad= \rho^{\frac{2q_1}{q}}\int_{-\infty}^{+\infty}(w(x_1,t_1))^{q_1}dt_1.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} &\le r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\int_{B_r(x)}||\mu(x_1,.)||_{L^{q_1}(\mathbb{R})}dx_1\\
&= r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\mu_2(B_r(x)).
\end{aligned}$$

Consequently, since (4.8.29) we derive (4.2.50) and (4.2.51).
We also obtain (4.2.52) from [1, Theorem 3.1]. ∎

4.8.3 Global estimates in $\mathbb{R}^N\times(0,\infty)$ and $\mathbb{R}^{N+1}$

Now, we present the proofs of Theorem 4.2.25 and 4.2.27.

Proof of Theorem 4.2.25 and Theorem 4.2.27. For any $n\ge 1$, it is easy to see that

i. $\mathbb{R}^N\backslash B_n(0)$ satisfies uniformly $2-$thick with constants $c_0=\frac{\text{Cap}_p(B_{1/4}(z_0),B_2(0))}{\text{Cap}_p(B_1(0),B_2(0))}$, $z_0=(1/2,0,...,0)\in\mathbb{R}^N$ and $r_0=n$.

ii. for any $\delta\in(0,1)$, $B_n(0)$ is a $(\delta,2n\delta)-$ Reifenberg flat domain.

iii. $[\mathcal{A}]_{s_0}^n\le[\mathcal{A}]_{s_0}^\infty$.

Assume that $||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}<\infty$. Thus by Remark 4.2.26 we have

$$\mathbb{I}_2[|\omega|](x,t)<\infty \text{ for a.e } (x,t)\in\mathbb{R}^{N+1}. \tag{4.8.30}$$

In view of the proof of the Theorem 4.2.5 and applying Theorem 4.2.17 to $B_n(0)\times(-n^2,n^2)$ and with data $\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega$ for any $n\ge 2$, there exists a sequence renormalized solution $\{u_n\}$ (we will take its subsequence if need) of

$$\left\{\begin{array}{l}
(u_n)_t-div(A(x,t,\nabla u_n))=\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega \text{ in } B_n(0)\times(-n^2,n^2),\\
u_n=0 \quad\text{on } \partial B_n(0)\times(-n^2,n^2),\\
u_n(-n^2)=0 \text{ in } B_n(0),
\end{array}\right.$$

converging to a distribution solution u in $L^1_{\text{loc}}(\mathbb{R}; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ of 4.2.6 with data $\mu = \omega$ such that

$$\begin{aligned}|||\nabla u_n|||_{L^{p,s}(B_n(0)\times(-n^2,n^2))} &\leq c_1||\mathbb{M}_1[\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}|\omega|]||_{L^{p,s}(B_{2n}(0)\times(-n^2,n^2))} \\ &\leq c_1||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}.\end{aligned}$$

Here $c_1 = c_1(N, \Lambda_1, \Lambda_2, p, s)$ is not depending on n since $\frac{T_0}{r_0} = \frac{2n+(1+n^2)^{1/2}}{n} \approx 1$.
Using Fatou Lemma, we get estimate (4.2.38).
As above, we also obtain (4.2.39).
And similarly, we can prove Theorem 4.2.27 by this way.
This completes the proof of Theorem. ■

Remark 4.8.5 (sharpness) *The inequality* (4.2.41) *is in a sense optimal as follows :*

$$C^{-1}||\mathbb{M}_1[|\omega|]||_{L^q(\mathbb{R}^{N+1})} \leq |||\nabla\mathcal{H}_2| * |\omega|||_{L^q(\mathbb{R}^N\times(0,\infty))} \leq C||\mathbb{M}_1[|\omega|]||_{L^q(\mathbb{R}^{N+1})} \tag{4.8.31}$$

for every $q > 1$ where $C = C(N, q)$. Indeed, we have

$$\nabla\mathcal{H}_2(x,t) = -\frac{C_\alpha}{2}\frac{\chi_{(0,\infty)}(t)}{t^{(N+1)/2}}\exp(-\frac{|x|^2}{4t})\frac{x}{\sqrt{t}},$$

leads to

$$\frac{c_1^{-1}}{t^{\frac{N+1}{2}}}\chi_{t>0}\chi_{\frac{1}{2}\sqrt{t}\leq|x|\leq 2\sqrt{t}} \leq |\nabla\mathcal{H}_\alpha(x,t)| \leq \frac{c_1}{\max\{|x|, \sqrt{2|t|}\}^{N+1}}.$$

Immediately, we get

$$c_2^{-1}\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r} \leq |\nabla\mathcal{H}_2| * |\omega|(x,t) \leq c_2\mathbb{I}_1[\omega](x,t).$$

By Theorem 4.4.2, gives the right-hand side inequality of (4.8.31)*. So, it is enough to show that*

$$A := \int_{\mathbb{R}^{N+1}}\left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r}\right)^q dxdt \geq c_3||\mathbb{M}_1[\omega]||^q_{L^q(\mathbb{R}^{N+1})} \tag{4.8.32}$$

To do this, we take $r_k = (3/2)^k$ for $k \in \mathbb{Z}$,

$$\begin{aligned}&\left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r}\right)^q \\ &\qquad \geq c_4\sum_{k=-\infty}^{\infty}\left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}}\right)^q.\end{aligned}$$

We deduce that

$$A \geq c_4\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^{N+1}}\left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}}\right)^q dxdt.$$

For any k, put $y = x + \frac{7}{8}r_k$ and $s = t - \frac{25}{32}r_k^2$, so $B_{r_k}(x)\backslash B_{3r_k/4}(x) \supset B_{r_k/8}(y)$ and

$$\int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t - r_k^2, t - 9r_k^2/16)\right)}{r_k^{N+1}} \right)^q dxdt$$
$$\geq \int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s - 7r_k^2/32, t + 7r_k^2/32)\right)}{r_k^{N+1}} \right)^q dyds.$$

Consequently,

$$A \geq c_4 \int_{\mathbb{R}^{N+1}} \sum_{k=-\infty}^{\infty} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s - 7r_k^2/32, t + 7r_k^2/32)\right)}{r_k^{N+1}} \right)^q dyds.$$

It follows (4.8.32).

4.9 Quasilinear Riccati Type Parabolic Equations

4.9.1 Quasilinear Riccati Type Parabolic Equation in Ω_T

We provide below only the proof of Theorem 4.2.30, 4.2.32 and 4.2.33. The proof of Theorem 4.2.31 can be proceeded by a similar argument.

Proof of Theorem 4.2.30. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T)$ be as in the proof of Theorem 4.2.1. We have $|\mu_n|(\Omega_T) \leq |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$. Let $\sigma_n \in C_c^\infty(\Omega)$ be converging to σ in the narrow topology of measures and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ such that $||\sigma_n||_{L^1(\Omega)} \leq |\sigma|(\Omega)$. For $n_0 \in \mathbb{N}$, we prove that the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}$ and $\sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0, T, W_0^{1,1}(\Omega)) : |||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda\},$$

where $L^{\frac{N+2}{N+1},\infty}(\Omega_T)$ is Lorent space with norm

$$||f||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} := \sup_{0<|D|<\infty} \left(|D|^{-\frac{1}{N+2}} \int_{D\cap\Omega_T} |f| \right).$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0, T, W_0^{1,1}(\Omega))$ and convex. We consider a map $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0, T, W_0^{1,1}(\Omega))$ is the unique solution of

$$\begin{cases} u_t - div\,(A(x, t, \nabla u)) = |\nabla v|^q + \mu_{n_0} & \text{in } \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\ u(0) = \sigma_{n_0}. \end{cases} \tag{4.9.1}$$

By Remark 4.3.2, we have

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq c_1 \left(|||\nabla v|^q||_{L^1(\Omega_T)} + |\mu_{n_0}|(\Omega_T) + ||\sigma_{n_0}||_{L^1(\Omega)} \right),$$

for some $c_1 = c_1(N, \Lambda_1, \Lambda_2)$. It leads to

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq c_1 \left(c_2 |\Omega_T|^{1-\frac{q(N+1)}{N+2}} |||\nabla v|||^q_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} + |\mu|(\Omega_T) + |\sigma|(\Omega) \right)$$
$$\leq c_1 \left(c_2 |\Omega_T|^{1-\frac{q(N+1)}{N+2}} \Lambda^q + |\mu|(\Omega_T) + |\sigma|(\Omega) \right),$$

for some $c_2 = c_2(N, q) > 0$. Thus, we now suppose that

$$|\Omega_T|^{-1+\frac{q'}{N+2}} \left(|\mu|(\Omega_T) + |\sigma|(\Omega) \right) \leq (2c_1)^{-q'} c_2^{-\frac{1}{q-1}},$$

then

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda := 2c_1(|\mu|(\Omega) + |\sigma|(\Omega)),$$

which implies that S is well defined.
Now we show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}_\Lambda$ such that v_n converges strongly in $L^1(0, T, W_0^{1,1}(\Omega))$ to a function $v \in \mathbf{E}_\Lambda$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^1(0, T, W_0^{1,1}(\Omega))$. We have

$$\begin{cases} (u_n)_t - \operatorname{div}\left(A(x, t, \nabla u_n)\right) = |\nabla v_n|^q + \mu_{n_0} & \text{in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_n(0) = \sigma_{n_0} \quad \text{in } \Omega, \end{cases} \tag{4.9.2}$$

satisfied

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda, \quad |||\nabla v_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda.$$

Thus, $|\nabla v_n|^q \to |\nabla v|^q$ in $L^1(\Omega_T)$. Therefore, it is easy to see that we get $u_n \to S(v)$ in $L^1(0, T, W_0^{1,1}(\Omega))$ by Theorem 4.3.6.
Next we show that S **is pre-compact**. Indeed if $\{u_n\} = \{S(v_n)\}$ is a sequence in $S(\mathbf{E}_\Lambda)$. By Proposition 4.3.5, there exists a subsequence of $\{u_n\}$ converging to some u in $L^1(0, T, W_0^{1,1}(\Omega))$. Consequently, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$ this means : the problem (4.2.53) has a solution with data μ_{n_0}, σ_{n_0}.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution u_n of

$$\begin{cases} (u_n)_t - div\left(A(x, t, \nabla u_n)\right) = |\nabla u_n|^q + \mu_n & \text{in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u_n(0) = \sigma_n, \end{cases} \tag{4.9.3}$$

which satisfies

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq 2c_1(|\mu|(\Omega) + |\sigma|(\Omega)).$$

Thanks to Proposition 4.3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0, T, W_0^{1,1}(\Omega))$. So, $|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq 2c_1(|\mu|(\Omega) + |\sigma|(\Omega))$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable. It follows the results by Proposition 4.3.5 and Theorem 4.3.6. ■

Proof of Theorem 4.2.32. Case a. A is linear operator. By Theorem 4.2.22, there exist $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that Ω is (δ, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \leq \delta$ for some R_0 and a sequence $\{u_n\}_n$ as distribution solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x,t,\nabla u_1)) = \mu \quad \text{in } \Omega_T, \\ u_1 = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_1(0) = \sigma \quad \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} (u_{n+1})_t - \operatorname{div}(A(x,t,\nabla u_{n+1})) = |\nabla u_n|^q + \mu \quad \text{in } \Omega_T, \\ u_{n+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{n+1}(0) = \sigma \quad \text{in } \Omega, \end{cases}$$

which satisfy

$$[|\nabla u_{n+1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq c_1 [|\nabla u_n|^q + \omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 0 \tag{4.9.4}$$

where $u_0 \equiv 0$ and constant c_1 depends only on $N, \Lambda_1, \Lambda_2, q$ and $T_0/R_0, T_0$. Moreover, if $\sigma \in L^1(\Omega)$ then $\{u_n\}$ is the sequence of renormalized solutions.
i. Suppose

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq (q-1)^{\frac{1}{q}} (qc_1 2^{q-1})^{-\frac{1}{q-1}}, \tag{4.9.5}$$

we prove that

$$[|\nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 1. \tag{4.9.6}$$

Indeed, clearly, we have (4.9.6) when $n = 1$. Now assume that (4.9.6) is true with $n = m$, that is,

$$[|\nabla u_m|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

From (4.9.4) we obtain

$$\begin{aligned} [|\nabla u_{m+1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} &\leq c_1 [|\nabla u_m|^q + \omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\leq c_1 2^{q-1} \left([|\nabla u_m|^q]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} + [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \right) \\ &\leq c_1 2^{q-1} \left(\left(\frac{qc_1 2^{q-1}}{q-1} \right)^q [\omega]^{q(q-1)}_{\mathfrak{M}^{\mathcal{G}_1,q'}} + 1 \right) [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\leq \frac{qc_1 2^{q-1}}{q-1} [\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}. \end{aligned}$$

Here, the last inequality is obtained by using (4.9.5). So, (4.9.6) is also true with $n = m+1$. Thus, (4.9.6) is true for all $n \geq 1$.
ii. Clearly, $u_{n+1} - u_n$ is the unique renormalized solution of

$$\begin{cases} u_t - div\,(A(x,t,\nabla u)) = |\nabla u_n|^q - |\nabla u_{n-1}|^q \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{4.9.7}$$

So, we have

$$[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq c_1[|\nabla u_n|^q - |\nabla u_{n-1}|^q]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}} \quad \forall n \geq 1.$$

Since, $|s_1^q - s_2^q| \leq q|s_1 - s_2|(\max\{s_1, s_2\})^{q-1}$ for any $s_1, s_2 \geq 0$ and using Holder inequality, we get

$$\begin{aligned}[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} &\leq c_1 q^q \left[|\nabla u_n - \nabla u_{n-1}|^q\right]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \left[(\max\{|\nabla u_n|, |\nabla u_{n-1}|\})^q\right]^{q-1}_{\mathfrak{M}^{\mathcal{G}_1,q'}} \\ &\leq c_1 q^q \left[|\nabla u_n - \nabla u_{n-1}|^q\right]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \left([|\nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} + [|\nabla u_{n-1}|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}}\right)^{q-1}\end{aligned}$$

which follows from (4.9.6),

$$[|\nabla u_{n+1} - \nabla u_n|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq C\left[|\nabla u_n - \nabla u_{n-1}|^q\right]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \ \forall\ n \geq 1$$

where

$$C = c_1 q^q \left(\frac{qc_1 2^q}{q-1}\right)^{q-1} [\omega]^{q(q-1)}_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

Hence, if $C < 1$ then u_n converges to $u = u_1 + \sum_{n=1}^{\infty}(u_{n+1} - u_n)$ in $L^q(0, T, W_0^{1,q}(\Omega))$ and satisfied

$$[|\nabla u|^q]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \frac{qc_1 2^{q-1}}{q-1}[\omega]^q_{\mathfrak{M}^{\mathcal{G}_1,q'}}.$$

Note that $C < 1$ is equivalent to

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq (c_1 q^q)^{-\frac{1}{q(q-1)}} \left(\frac{qc_1 2^q}{q-1}\right)^{-\frac{1}{q}}$$

Combining this with (4.9.5) and using Theorem 4.3.6, we conclude that the problem (4.2.53) has a distribution solution u (a renormalized if $\sigma \in L^1(\Omega)$) , if

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_1,q'}} \leq \min\left\{(q-1)^{\frac{1}{q}}(qc_1 2^{q-1})^{-\frac{1}{q-1}}, (c_1 q^q)^{-\frac{1}{q(q-1)}}\left(\frac{qc_1 2^q}{q-1}\right)^{-\frac{1}{q}}\right\}.$$

Next, we will prove **Case b.** and **Case c.**.
Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \sigma_n \in C_c^\infty(\Omega)$ be as in the proof of Theorem 4.2.1. We have $|\mu_n| \leq \varphi_n * |\mu|, |\sigma_n| \leq \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}$, $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively. Set $\omega_n = |\mu_n| + |\sigma_n| \otimes \delta_{\{t=0\}}$ and $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$.
Case b. For $n_0 \in \mathbb{N}$, $\varepsilon > 0$, we prove that the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0, T, W_0^{1,1}(\Omega)) : [|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)} \leq \Lambda\}.$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0, T, W_0^{1,1}(\Omega))$ and convex. We consider a map $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0, T, W_0^{1,1}(\Omega))$ is the unique solution of problem (4.9.1). By Theorem 4.2.22, there exist $\delta = \delta(N, \Lambda_1, \Lambda_2, q+\varepsilon) \in (0, 1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that Ω is (δ, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \leq \delta$ for some R_0 we have

$$[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \leq c_2[|\nabla v|^q + \omega_{n_0}]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}},$$

where $c_2 = c_2(N, \Lambda_1, \Lambda_2, q+\varepsilon, T_0/R_0, T_0)$. By Remark 4.4.33, we deduce that

$$[|\nabla v|^q]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_3 [|\nabla v|^{q+\varepsilon}]^{\frac{q}{q+\varepsilon}}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}},$$

where a constant c_3 depends on $N, q+\varepsilon$.
Thus,

$$\begin{aligned}
[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} &\le c_2 \left([|\nabla v|^q]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\
&\le c_2 \left(c_3 [|\nabla u|^{q+\varepsilon}]^{\frac{q}{q+\varepsilon}}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\
&\le c_2 \left(c_3 \Lambda^{\frac{q}{q+\varepsilon}} + [\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}\right)^{q+\varepsilon} \\
&\le \Lambda,
\end{aligned}$$

provided that $[\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4 := 2^{-q'} c_2^{-\frac{q'}{q+\varepsilon}} c_3^{-\frac{1}{q-1}}$ and $\Lambda := 2^{q+\varepsilon} c_2 [\omega_{n_0}]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}$.
which implies that S is well defined with $[\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4$.
Now we show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}_\Lambda$ such that v_n converges strongly in $L^1(0,T,W_0^{1,1}(\Omega))$ to a function $v \in \mathbf{E}_\Lambda$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$. We have u_n satisfied (4.9.2) and

$$[|\nabla u_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le \Lambda, \quad [|\nabla v_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le \Lambda.$$

In particular, $||\nabla v_n||_{L^{q+\varepsilon}(\Omega_T)} \le \Lambda \mathrm{Cap}_{\mathcal{G}_1,(q+\varepsilon)'}(\overline{\Omega}_T)$ for all n. Thus, $|\nabla v_n|^q \to |\nabla v|^q$ in $L^1(\Omega_T)$. Therefore, it is easy to see that we get $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$ by Theorem 4.3.6. On the other hand, S **is pre-compact**. Therefore, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. Hence the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
Thanks to Corollary 4.4.39 and Remark 4.4.40 we get

$$[\omega_n]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_5 [\omega]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \quad \forall \;\; n \in \mathbb{N}, \tag{4.9.8}$$

where $c_5 = c_5(N, q+\varepsilon, T_0)$.
Assume that $[\omega]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4 c_5^{-1}$. So $[\omega_n]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le c_4$ for all n.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution u_n of problem (4.9.3) which satisfies

$$[|\nabla u_n|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le 2^{q+\varepsilon} c_2 [\omega_n]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}} \le 2^{q+\varepsilon} c_2 c_5^{q+\varepsilon} [\omega]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}}.$$

By Proposition 4.3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0,T,W_0^{1,1}(\Omega))$. So, $[|\nabla u|^{q+\varepsilon}]_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)} \le 2^{q+\varepsilon} c_2 c_5^{q+\varepsilon} [\omega]^{q+\varepsilon}_{\mathfrak{M}^{\mathcal{G}_1,(q+\varepsilon)'}(\Omega_T)}$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable. It follows the result by Proposition 4.3.5 and Theorem 4.3.6.

Case c. For $n_0 \in \mathbb{N}$. We prove that the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \{u \in L^1(0,T,W_0^{1,1}(\Omega)) : |||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \le \Lambda\},$$

where $L^{(N+2)(q-1),\infty}(\Omega_T)$ is Lorent space with norm

$$||f||_{L^{(N+2)(q-1),\infty}(\Omega_T)} := \sup_{0<|D|<\infty}\left(|D|^{-1+\frac{1}{(N+2)(q-1)}}\int_{D\cap\Omega_T}|f|dxdt\right).$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex. We consider a map $S:\mathbf{E}_\Lambda\to\mathbf{E}_\Lambda$ defined for each $v\in\mathbf{E}_\Lambda$ by $S(v)=u$, where $u\in L^1(0,T,W_0^{1,1}(\Omega))$ is the unique solution of problem (4.9.1). By Theorem 4.2.20, there exist $\delta=\delta(N,\Lambda_1,\Lambda_2,q)\in(0,1)$ and $s_0=s_0(N,\Lambda_1,\Lambda_2)>0$ such that Ω is (δ,R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0}\le\delta$ for some R_0 we have

$$\begin{aligned}|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} &\le c_6||\mathbb{M}_1[|\nabla v|^q+\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}\\ &\le c_6\left(||\mathbb{M}_1[|\nabla v|^q]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}+||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}\right),\end{aligned}$$

where $c_6=c_6(N,\Lambda_1,\Lambda_2,q,T_0/R_0)$ and $T_0=\operatorname{diam}(\Omega)+T^{1/2}$.
By Proposition 4.4.4 we have

$$\begin{aligned}||\mathbb{M}_1[|f|^q]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})} &\le c_7||\mathbb{I}_1[|f|^q]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})}\\ &\le c_8||f||^q_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1})}\quad\forall f\in L^{(N+2)(q-1),\infty}(\mathbb{R}^{n+1}),\end{aligned}$$

where a constant c_8 only depends on N,q. Thus,

$$\begin{aligned}|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} &\le c_6\left(c_8|||\nabla v|||^q_{L^{(N+2)(q-1),\infty}(\Omega_T)}+||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}\right)\\ &\le c_6\left(c_8\Lambda^q+||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}\right),\end{aligned}$$

which implies that S is well defined with $||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}\le c_9:=(2c_6)^{-q'}c_8^{-\frac{1}{q-1}}$ and $\Lambda:=2c_6||\mathbb{M}_1[\omega_{n_0}]||_{L^{(N+2)(q-1),\infty}(\Omega_T)}$.
As in **Case b** we can show that $S:\mathbf{E}_\Lambda\to\mathbf{E}_\Lambda$ is continuous and $S(\mathbf{E}_\Lambda)$ is pre-compact, thus by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. Hence the problem (4.2.53) has a solution with data $\mu=\mu_{n_0},\sigma=\sigma_{n_0}$.
To continue, we need to show that

$$\begin{aligned}&||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}\\ &\quad\le c_{10}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}+c_{10}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)},\end{aligned}\tag{4.9.9}$$

for every $n\ge k_0$. Where k_0 is a constant large enough and $c_{10}=c_{10}(N,q)$ Indeed, we have $\mathbb{M}_1[\omega_n]\le c_{11}\mathbb{I}_1[\varphi_n*|\mu|]+c_{11}\mathbb{I}_1[(\varphi_{1,n}*|\sigma|)\otimes\delta_{\{t=0\}}]$. Thus, by Proposition 4.4.19 we deduce

$$\begin{aligned}&||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}\\ &\le c_{11}||\mathbb{I}_1[\varphi_n*|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}+c_{12}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[\varphi_{1,n}*|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)}\\ &=c_{11}||\varphi_n*\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}+c_{12}||\varphi_{1,n}*\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)}\\ &\to c_{11}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}+c_{12}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)}\quad\text{as } n\to\infty.\end{aligned}$$

It implies (4.9.9).
Now we assume that

$$||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}, ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \leq c_9(2c_{10})^{-1},$$

then $||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} \leq c_9$ for all $n \geq k_0$. Consequently, there exists a renormalized solution u_n of problem (4.9.3) satisfied

$$\begin{aligned}|||\nabla u_n|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} &\leq 2c_6||\mathbb{M}_1[\omega_n]||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \\ &\leq 2c_6c_{10}||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + 2c_6c_{10}||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} =: C\end{aligned}$$

for any $n \geq k_0$. Thanks to Proposition 4.3.5, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0,T,W_0^{1,1}(\Omega))$. So, $|||\nabla u|||_{L^{(N+2)(q-1),\infty}(\Omega_T)} \leq C$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable.
It follows the result by Proposition 4.3.5 and Theorem 4.3.6. This completes the proof. ■

Proof of Theorem 4.2.33. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \sigma_n \in C_c^\infty(\Omega)$ be as in the proof of Theorem 4.2.1. We have $|\mu_n| \leq \varphi_n * |\mu|, |\sigma_n| \leq \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}$, $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively. We can assume that $\text{supp}(\mu_n) \subset (\Omega' + B_{d/4}(0)) \times [0,T]$ and $\text{supp}(\sigma_n) \subset \Omega' + B_{d/4}(0)$ for any $n \in \mathbb{N}$. Set $\omega_n = |\mu_n| + |\sigma_n| \otimes \delta_{\{t=0\}}$ and $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$.
First, we prove that the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$ for $n_0 \in \mathbb{N}$. By Corollary 4.4.39 and Remark 4.4.40, we have

$$[\omega_n]_{\mathfrak{M}^{\mathcal{G}_{1},q'}} \leq c_1\varepsilon_0 \;\forall n \in \mathbb{N}, \tag{4.9.10}$$

where $c_1 = c_1(N,q,T_0)$ and $\varepsilon_0 = [\omega]_{\mathfrak{M}^{\mathcal{G}_{1},q'}}$. By Proposition 4.4.36 and Remark 4.4.37, we have

$$\mathbb{I}_1^{2T_0,\delta}\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_n]\right)^q\right] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_1^{2T_0,\delta}[\omega_n] \quad \text{a.e in } \mathbb{R}^{N+1} \text{ and} \tag{4.9.11}$$

$$\mathbb{I}_2[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_n]\right)^q] \leq c_2\varepsilon_0^{q-1}\mathbb{I}_2[\omega_n] \quad \text{a.e in } \mathbb{R}^{N+1}, \tag{4.9.12}$$

for any $n \in \mathbb{N}$, where $c_2 = c_2(N,\delta,q,T_0)$ and $0 < \delta < 1$. We set

$$\mathbf{E}_\Lambda = \{u \in L^1(0,T,W_0^{1,1}(\Omega)) : |\nabla u| \leq \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\}.$$

Clearly, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex.
We consider a map $S : \mathbf{E}_\Lambda \to L^1(0,T,W_0^{1,1}(\Omega))$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0,T,W_0^{1,1}(\Omega))$ is the unique renormalized solution of problem (4.9.1). We will show that $S(\mathbf{E}_\Lambda)$ is subset of $\mathbf{E}_\Lambda$ for some $\Lambda > 0$ and ε_0 small enough.
We have

$$|\nabla v| \leq \Lambda\mathbb{I}_1[\omega_{n_0}]. \tag{4.9.13}$$

In particular, $|||\nabla v|||_{L^\infty(\Omega_{d/2}\times(0,T))} \leq \Lambda(N+1)^{-1}(d/2)^{-N-1}\omega_{n_0}(\overline{\Omega_T})$, where $\Omega_{d/2} = \{x \in \Omega : d(x,\partial\Omega) \leq d/2\}$.
From (4.9.11) and (4.9.12) lead to

$$\mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q] \leq \Lambda^q\mathbb{I}_1^{2T_0,\delta}\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\right)^q\right] \leq c_2\Lambda^q\varepsilon_0^{q-1}\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}] \text{ and}$$

$$\mathbb{I}_2[|\nabla v|^q] \leq \Lambda^q\mathbb{I}_2\left[\left(\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}]\right)^q\right] \leq c_2\Lambda^q\varepsilon_0^{q-1}\mathbb{I}_2[\omega_{n_0}].$$

Clearly, from [27, Theorem 1.2], we have for any $Q_r(x,t) \subset\subset \Omega \times (-\infty, T)$ with $r \le r_0$

$$\begin{aligned}
|\nabla u(x,t)| &\le c_3 \fint_{Q_r(x,t)} |\nabla u| dyds + c_3 \mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q + \omega_{n_0}](x,t) \\
&\le c_3 \fint_{Q_r(x,t)} |\nabla u| dyds + c_3 \mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q](x,t) + c_3 \mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t) \\
&\le c_3 \fint_{Q_r(x,t)} |\nabla u| dyds + c_3 \left(c_2 \Lambda^q \varepsilon_0^{q-1} + 1\right) \mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t), \qquad (4.9.14)
\end{aligned}$$

where $c_3 = c_3(N, \Lambda_1)$ and $r_0 = r_0(N, \Lambda_1, \Lambda_2, \Lambda_3, \beta) > 0$.
Since $|||\nabla u|||_{L^1(\Omega_T)} \le c_4 T_0 \left(|||\nabla v|||^q_{L^q(\Omega_T)} + \omega_{n_0}(\overline{\Omega_T})\right)$, for any $(x,t) \in (\Omega \backslash \Omega_{d/4}) \times (-\infty, T)$ where $\Omega_{d/4} = \{x \in \Omega : d(x, \partial\Omega) \le d/4\}$,

$$\begin{aligned}
\frac{1}{|Q_{d_0}(x,t)|} \int_{Q_{d_0}(x,t)} |\nabla u| dyds &\le c_5 d_0^{-N-2} T_0 \left(|||\nabla v|||^q_{L^q(\Omega_T)} + \omega_{n_0}(\overline{\Omega_T})\right) \\
&\le c_6 \mathbb{I}_1^{2T_0,\delta}[|\nabla v|^q + \omega_{n_0}](x,t) \\
&\le c_6 \left(c_2 \Lambda^q \varepsilon_0^{q-1} + 1\right) \mathbb{I}_1^{2T_0,\delta}|[\omega_{n_0}](x,t), \qquad (4.9.15)
\end{aligned}$$

where $d_0 = \min\{d/8, r_0\}$ and $c_6 = c_6(N, p, \Lambda_1, \Lambda_2, T_0/d_0)$.
By regularity theory, we have

$$||\nabla u||_{L^\infty(\Omega_{d/4} \times (0,T))} \le c_7 (||u||_{L^\infty(\Omega_{d/2} \times (0,T))} + |||\nabla v|^q||_{L^\infty(\Omega_{d/2} \times (0,T))}),$$

where $c_7 = c_7(N, \Lambda_1, \Lambda_2, \Lambda_3, \Omega, T)$.
a. Estimate $|||\nabla v|^q||_{L^\infty(\Omega_{d/2} \times (0,T))}$. Thanks to (4.9.13),

$$|||\nabla v|^q||_{L^\infty(\Omega_{d/2} \times (0,T))} \le \left(\Lambda (d/2)^{-N-1} (\omega_{n_0}(\overline{\Omega_T}))\right)^q.$$

Since $\omega_{n_0}(\overline{\Omega_T}) \le c_1 \varepsilon_0 \text{Cap}_{\mathcal{G}_1, q'}(\tilde{Q}_{T_0}(x_0, t_0)) = c_8(N, q, p, T_0)\varepsilon_0$ with $(x_0, t_0) \in \Omega_T$, thus

$$|||\nabla v|^q||_{L^\infty(\Omega_{d/2} \times (0,T))} \le c_9 \Lambda^q \varepsilon_0^{q-1} \mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t) \quad \forall (x,t) \in \Omega_T,$$

where $c_9 = c_9(N, \Lambda_1, \Lambda_2, \Lambda_3, q, d, \Omega, T)$.
b. Estimate $||u||_{L^\infty(\Omega_{d/2})}$. By Theorem 4.2.1 we have

$$|u(x,t)| \le c_{10} \mathbb{I}_2[|\nabla v|^q + \omega_{n_0}](x,t) \quad \forall (x,t) \in \Omega_T,$$

where $c_{10} = c_{10}(N, \Lambda_1, \Lambda_2)$. Thus,

$$\begin{aligned}
|u(x,t)| &\le c_{10} \mathbb{I}_2[|\nabla v|^q](x,t) + c_{10} \mathbb{I}_2[\omega_{n_0}](x,t) \\
&\le c_{10} \left(c_2 \Lambda^q \varepsilon_0^{q-1} + 1\right) \mathbb{I}_2[\omega_{n_0}](x,t),
\end{aligned}$$

which implies

$$\begin{aligned}
||u||_{L^\infty(\Omega_{d/2} \times (0,T))} &\le c_{11} \left(c_2 \Lambda^q \varepsilon_0^{q-1} + 1\right) d^{-N} \omega_{n_0}(\overline{\Omega_T}) \\
&\le c_{12} \left(c_2 \Lambda^q \varepsilon_0^{q-1} + 1\right) \mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t) \quad \forall (x,t) \in \Omega_T,
\end{aligned}$$

where $c_{12} = c_{12}(N, \Lambda_1, \Lambda_2, \Lambda_3, q, T_0/d)$. Therefore,

$$||\nabla u||_{L^\infty(\Omega_{d/4}\times(0,T))} \leq c_{13}\left(c_{14}\Lambda^q\varepsilon_0^{q-1}+1\right)\inf_{(x,t)\in\Omega_T}\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t). \tag{4.9.16}$$

where $c_{13} = c_{13}(N, \Lambda_1, \Lambda_2, \Lambda_3, q, d, \Omega, T)$.
Finally from (4.9.15) (4.9.16) and (4.9.14) we get for all $(x,t) \in \Omega_T$

$$|\nabla u(x,t)| \leq c_{14}\left(c_{15}\Lambda^q\varepsilon_0^{q-1}+1\right)\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t).$$

where $c_{14} = c_{14}(N, \Lambda_1, \Lambda_2, \Lambda_3, q, d, \Omega, T)$ and $c_{15} = c_{15}(N, \delta, q)$.
So, we suppose that $\Lambda = 2c_{14}$ and $\varepsilon_0 \leq c_{15}^{-\frac{1}{q-1}}(2c_{14})^{-\frac{q}{q-1}}$, it is equivalent to (4.2.61), (4.2.62) holding for some $C > 0$. Then for any $(x,t) \in \Omega_T$

$$|\nabla u(x,t)| \leq \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_{n_0}](x,t),$$

and S is well defined.
On the other hand, we can see that $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ is continuous and $S(E)$ is pre-compact under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$.
Thus, by Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. This means : the problem (4.2.53) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.
Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution u_n of problem (4.9.3) which satisfies

$$|\nabla u_n(x,t)| \leq \Lambda\mathbb{I}_1^{2T_0,\delta}[\omega_n](x,t)\ \forall\ (x,t) \in \Omega_T.$$

Since $\mathbb{I}_1^{2T_0,\delta}[\omega_n](x,t) \leq \varphi_n * \mathbb{I}_1^{2T_0,\delta}[|\mu|](x,t) + \varphi_{1,n} * (\mathbb{I}_1^{2T_0,\delta}[|\sigma|\otimes\delta_{\{t=0\}}](.,t))(x) =: A_n(x,t)$ and A_n converges to $\mathbb{I}_1^{2T_0,\delta}[|\mu|] + \mathbb{I}_1^{2T_0,\delta}[|\sigma|\otimes\delta_{\{t=0\}}]$ in $L^q(\mathbb{R}^{N+1})$, thus $|\nabla u_n|^q$ is equi-integrable. As in the proof of Theorem 4.2.32, we get the result by using Proposition 4.3.5 and Theorem 4.3.6. This completes the proof. ■

4.9.2 Quasilinear Riccati Type Parabolic Equation in $\mathbb{R}^N \times (0,\infty)$ and $\mathbb{R}^{N+1}$

In this subsection, we provide the proofs of Theorem 4.2.37 and 4.2.38. In the same way, we can prove Theorem 4.2.36.
Proof of Theorem 4.2.37. As in the proof of Theorem 4.2.25 and Theorem 4.2.27, we can apply Theorem 4.2.32 to obtain : there exists a constant $c_1 = c_1(N, \Lambda_1, \Lambda_2, q)$ that if $[A]_{s_0}^\infty \leq \delta$ and (4.2.64) holds with constant c_1 then we can find a sequence of renormalized solutions $\{u_{n_k}\}$ of

$$\begin{cases} (u_{n_k})_t - div(A(x,t,\nabla u_{n_k})) = |\nabla u_{n_k}|^q + \chi_{D_{n_k-1}}\omega \text{ in } D_{n_k}, \\ u_{n_k} = 0 \quad \text{on } \partial B_{n_k}(0)\times(-n_k^2, n_k^2), \\ u_{n_k}(-n_k^2) = 0 \ \text{ on } B_{n_k}(0). \end{cases}$$

converging to some u in $L^1_{\text{loc}}(\mathbb{R}; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ and satisfying

$$|||\nabla u_{n_k}|||_{L^{(q-1)(N+2),\infty}(D_{n_k})} \leq c_2 ||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})},$$

for some $c_2 = c_2(N, \Lambda_1, \Lambda_2, q)$, where $D_n = B_n(0) \times (-n^2, n^2)$. It follows $|\nabla u_{n_k}|^q \to |\nabla u|^q$ in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. Thus, u is a distribution solution of (4.2.55) which satisfies (4.2.63). Furthermore, if $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u_{n_k} = 0$ in $B_{n_k}(0) \times (-n_k^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distribution solution to (4.2.54).

■

Proof of Theorem 4.2.38. Let $\omega_n = \varphi_n * (\chi_{D_{n-1}}\omega)$ for any $n \geq 2$. We have $\mu_n \in C_c^\infty(\mathbb{R}^{N+1})$ with $\text{supp}(\omega_n) \subset D_n$ and $\omega_n \to \omega$ weakly in $\mathfrak{M}(\mathbb{R}^{N+1})$.
According to Corollary 4.4.39 and Remark 4.4.40, we have

$$[\omega_n]_{\mathfrak{M}^{\mathcal{H}_1,q'}} \leq c_1 \varepsilon_0 \quad \forall n \in \mathbb{N}$$

where $c_1 = c_1(N, q)$ and $[\omega]_{\mathfrak{M}^{\mathcal{H}_1,q'}} \leq \varepsilon_0$. Thus, thanks to Theorem 1.3 we get

$$\mathbb{I}_1\left[(\mathbb{I}_1[\omega_n])^q\right] \leq c_2 \varepsilon_0^{q-1} \mathbb{I}_1[\omega_n] \quad \text{and} \tag{4.9.17}$$

$$\mathbb{I}_2\left[(\mathbb{I}_1[\omega_n])^q\right] \leq c_2 \varepsilon_0^{q-1} \mathbb{I}_2[\omega_n] \quad \forall n \in \mathbb{N}, \tag{4.9.18}$$

where $c_2 = c_2(N, q, c_1)$.
We fix $n_0 \in \mathbb{N}$, put :

$$\mathbf{E}_\Lambda = \left\{ u \in L^1(-n_0^2, n_0^2, W_0^{1,1}(B_{n_0}(0))) : |\nabla u| \leq \Lambda \mathbb{I}_1[\omega_{n_0}] \ \text{ in } \ B_{n_0/4}(0) \times (-n_0^2, n_0^2) \right\}.$$

By using estimate (4.5.8) in Remark 4.5.3, we can apply the argument of the proof of Theorem 4.2.9, with problem (4.6.9) replaced by

$$\begin{cases} u_t - div\,(A(t, \nabla u)) = \chi_{B_{n_0/4}(0)\times(-n_0^2,n_0^2)} |\nabla v|^q + \omega_{n_0} \text{ in } \ D_{n_0}, \\ u = 0 \quad \text{ on } \quad \partial B_{n_0}(0) \times (-n_0^2, n_0^2), \\ u(-n_0^2) = 0 \quad \text{in} \quad B_{n_0}(0), \end{cases}$$

to obtain : the operator S (in the proof of Theorem 4.2.9) has a fixed point on $\mathbf{E}_\Lambda$ for some $\Lambda = \Lambda(N, \Lambda_1, \Lambda_2, q) > 0$ and $\varepsilon_0 = \varepsilon_0(N, \Lambda_1, \Lambda_2, q) > 0$. Therefore, for any $n \in \mathbb{N}$ there exists a solution u_n of problem

$$\begin{cases} (u_n)_t - div\,(A(t, \nabla u_n)) = \chi_{B_{n/4}(0)\times(-n^2,n^2)} |\nabla u_n|^q + \omega_n \text{ in } \ D_n, \\ u_n = 0 \quad \text{ on } \quad \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{in} \quad B_n(0), \end{cases}$$

which satisfies

$$|\nabla u_n(x,t)| \leq \Lambda \mathbb{I}_1[\omega_n](x,t) \quad \forall (x,t) \in B_{n/4}(0) \times (-n^2, n^2).$$

Moreover, combining this with (4.9.18) and Theorem 4.2.1 we also obtain

$$\begin{aligned} |u_n(x,t)| &\leq K \mathbb{I}_2 \left[\chi_{B_{n/4}(0)\times(-n^2,n^2)} |\nabla u_n|^q + |\omega_n| \right](x,t) \\ &\leq K \Lambda^q \mathbb{I}_2\left[(\mathbb{I}_1[|\omega_n|])^q\right] + K \mathbb{I}_2\left[|\omega_n|\right](x,t) \\ &\leq c_3 \mathbb{I}_2\left[|\omega_n|\right](x,t) \\ &\leq c_3 \varphi_n * \mathbb{I}_2\left[|\chi_{D_{n-1}}\omega|\right](x,t), \end{aligned}$$

for any $(x,t) \in B_n(0) \times (-n^2, n^2)$.
Since $\mathbb{I}_2[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$, thus $\sup_n \int_{D_m} \chi_{D_n} |u_n|^{q_0} dxdt < \infty$ for all $m \in \mathbb{N}$, $1 < q_0 < \frac{N+2}{N}$.
In addition, since $\mathbb{I}_1[\omega] \in L^q_{\text{loc}}(\mathbb{R}^{N+1})$, thus $\varphi_n * \mathbb{I}_1\left[|\chi_{D_{n-1}}\omega|\right] \to \mathbb{I}_1[\omega]$ in $L^q_{\text{loc}}(\mathbb{R}^{N+1})$ and $\{\chi_{B_{n/4}(0)\times(-n^2,n^2)}|\nabla u_n|^q\}$ is equi local integrable in $\mathbb{R}^{N+1}$.
Therefore, we can apply Corollary 4.3.18 to obtain : $u_n \to u$ in $L^1_{\text{loc}}(\mathbb{R}; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ (we will take its subsequence if need) and u satisfies (4.2.66). Also, $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1_{loc}(\mathbb{R}^{N+1})$.
Finally, we can conclude that u is a distribution solution of problem (4.2.65). Note that the assumption $[\omega]_{\mathfrak{M}^{\mathcal{H}_1,q'}} \le \varepsilon_0$ is equivalent to (4.2.67) holding with $C = \varepsilon_0$.
Furthermore, if $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u_n = 0$ in $B_n(0) \times (-n^2, a_n)$ where $\text{supp}(\omega_n) \subset \mathbb{R}^N \times (a_n, \infty)$ and $a_n \to 0^-$ as $n \to \infty$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distribution solution to (4.2.68).
This completes the proof of the Theorem. ■

4.10 Appendix

Proof of the Remark 4.2.7. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ if $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then for any $0 < \beta \le \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$. Indeed, by Remark 4.4.28 we have $\mathbb{I}_\alpha[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$.
Take $0 < \beta \le \alpha$ and $0 < s < \frac{N+2}{N+2-\beta}$. For $R > 0$, by Proposition 4.4.4 we have $\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$. Thus,

$$\begin{aligned}
&\int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\beta[\omega](x,t))^s \, dxdt \\
&\le c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)^c}\omega](x,t)\right)^s dxdt \\
&\le c_1 \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + c_1 R^{-s(\alpha-\beta)} \int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\alpha[\omega](x,t))^s \, dxdt \\
&< \infty.
\end{aligned}$$

For $0 < \beta < \alpha < N+2$, we consider

$$\omega(x,t) = \sum_{k=4}^{\infty} \frac{a_k}{|\tilde{Q}_{k+1}(0,0) \backslash \tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)}(x,t),$$

where $a_k = 2^{n(N+2-\theta)}$ if $k = 2^n$ and $a_k = 0$ otherwise with $\theta \in (\beta, \alpha]$.
It is easy to see that $\mathbb{I}_\alpha[\omega] \equiv \infty$ and $\mathbb{I}_\beta[\omega] < \infty$ in $\mathbb{R}^{N+1}$. ■
Proof of the Remark 4.2.26. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, since $\mathbb{I}_2[\omega] \le c_1 I_1[I_1[\omega]]$ thus :
If $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s < N+2$, then by Proposition 4.4.4 in next section

$$||\mathbb{I}_2[\omega]||_{L^{\frac{s(N+1)}{N+2-s},\infty}(\mathbb{R}^{N+1})} \le c_1 ||\mathbb{I}_1[\omega]||_{L^{s,\infty}(\mathbb{R}^{N+1})} < \infty$$

If $\mathbb{I}_1[\omega] \in L^{N+2,\infty}(\mathbb{R}^{N+1})$, then by Theorem 4.4.3,

$$\mathbb{I}_2[\omega] \in L^{s_0}_{\text{loc}}(\mathbb{R}^{N+1}) \;\; \forall \; s_0 > 1$$

So, $\mathbb{I}_2[\omega] < \infty$ a.e in $\mathbb{R}^{N+1}$ if $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s \le N+2$.
For $s > N+2$, there exists $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\mathbb{I}_2[\omega] \equiv \infty$ in $\mathbb{R}^{N+1}$ and $\mathbb{I}_1[\omega] \in L^s(\mathbb{R}^{N+1})$. Indeed, consider

$$\omega(x,t) = \sum_{k=1}^{\infty} \frac{k^{N-1}}{|\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)}(x,t).$$

We have for $(x,t) \in \mathbb{R}^{N+1}$ and $n_0 \in \mathbb{N}$ with $n_0 > \log_2(\max\{|x|, \sqrt{2|t|}\})$

$$\begin{aligned}
\mathbb{I}_2[\omega](x,t) &\ge c_2 \sum_{n_0}^{\infty} \frac{\omega(\tilde{Q}_{2^n}(x,t))}{2^{nN}} \ge c_2 \sum_{n_0}^{\infty} \frac{\omega(\tilde{Q}_{2^{n-1}}(0,0))}{2^{nN}} \\
&\ge c_2 \sum_{n_0}^{\infty} \frac{\sum_{k=1}^{2^{n-1}-1} k^{N-1}}{2^{nN}} = c_2 \sum_{k=1}^{\infty} \left(\sum_{n_0}^{\infty} \chi_{k \le 2^{n-1}-1} \frac{1}{2^{nN}} \right) k^{N-1} \\
&\ge c_4 \sum_{k=n_0}^{\infty} k^{-1} = \infty.
\end{aligned}$$

On the other hand, for $s_1 > \frac{N+2}{2}$

$$\int_{\mathbb{R}^{N+1}} \omega^{s_1} dxdt = c_5 \sum_{k=1}^{\infty} \frac{k^{s(N-1)}}{((k+1)^{N+2} - k^{N+2})^{s_1-1}} \le c_6 \sum_{k=1}^{\infty} \frac{k^{s_1(N-1)}}{k^{(s_1-1)(N+1)}} < \infty,$$

since $(s_1-1)(N+1) - s_1(N-1) > 1$. Thus,

$$||\mathbb{I}_1[\omega]||_{L^s(\mathbb{R}^{N+1})} \le c_7 ||\omega||_{L^{\frac{s(N+2)}{N+2+s}}(\mathbb{R}^{N+1})} < \infty.$$

■

Proof of the Proposition 4.3.16. We will use an idea in [9, 10] to prove 4.3.14. For $S' \in W^{1,\infty}(\mathbb{R})$ with $S(0) = 0$, $S'' \ge 0$, $S'(\tau)\tau \ge 0$ for all $\tau \in \mathbb{R}$ and $||S'||_{L^\infty(\mathbb{R})} \le 1$ we have

$$\begin{aligned}
&-\int_D \eta_t S(u) dxdt + \int_D S'(u) A(x,t,\nabla u) \nabla \eta dxdt \\
&\qquad + \int_D S''(u) \eta A(x,t,\nabla u) \nabla u dxdt + \int_D S'(u) \eta L(u) dxdt = \int_D S'(u) \eta d\mu.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Lambda_2 \int_D S''(u) \eta |\nabla u|^2 dxdt \\
&\qquad + \int_D S'(u) \eta L(u) dxdt \le \Lambda_1 \int_D |\nabla u||\nabla \eta| dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u| dxdt.
\end{aligned}$$

a. We choose $S' \equiv \varepsilon^{-1} T_\varepsilon$ for $\varepsilon > 0$ and let $\varepsilon \to 0$ we will obtain

$$\int_D \eta |L(u)| dxdt \le \Lambda_1 \int_D |\nabla u||\nabla \eta| dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u| dxdt. \tag{4.10.1}$$

b. for $S'(u) = (1 - (|u|+1)^{-\alpha})\mathrm{sign}(u)$ for $\alpha > 0$ then

$$\int_D \frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt \le c_1 \left(\int_D |\nabla u||\nabla \eta| dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u|dxdt \right),$$

Using Holder's inequality, we have

$$\int_D |\nabla u||\nabla \eta| dxdt \le \frac{1}{2c_1}\int_D \frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt + c_2 \int_D (|u|+1)^{q_0}\eta dxdt + c_2 \int_D |\nabla \eta^{1/q_1}|^{q_1} dxdt.$$

Hence,

$$\int_D |\nabla u||\nabla \eta| dxdt + \int_D \frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt \le c_3 B. \tag{4.10.2}$$

c. for $S'(u) = \frac{-k+\delta+|u|}{2\delta}\mathrm{sign}(u)\chi_{k-\delta<|u|<k+\delta} + \mathrm{sign}(u)\chi_{|u|\ge k+\delta}$, $0 < \delta \le k$ then

$$\frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta} |\nabla u|^2 \eta dxdt \le c_4 \left(\int_D |\nabla u||\nabla \eta| dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u|dxdt \right). \tag{4.10.3}$$

In particular,

$$\frac{1}{k}\int_D |\nabla T_k(u)|^2 \eta dxdt \le c_5 \left(\int_D |\nabla u||\nabla \eta| dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u|dxdt \right) \quad \forall k > 0. \tag{4.10.4}$$

Consequently, we deduce (4.3.14) from (4.10.1)-(4.10.4).
Next, take $\varphi \in C_c^\infty(D)$ and $S'(u) = \chi_{|u|\le k-\delta} + \frac{k+\delta-|u|}{2\delta}\chi_{k-\delta<|u|<k+\delta}$, $S(0) = 0$ we have

$$\begin{aligned}
&-\int_D \varphi_t \eta S(u) dxdt + \int_D S'(u)\eta A(x,t,\nabla u)\nabla \varphi dxdt + \int_D S'(u)\varphi A(x,t,\nabla u)\nabla \eta dxdt \\
&\quad - \frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta} \mathrm{sign}(u)\varphi\eta A(x,t,\nabla u)\nabla u dxdt + \int_D S'(u)\varphi\eta L(u) dxdt \\
&\quad = \int_D S'(u)\varphi\eta d\mu + \int_D \varphi\eta_t S(u) dxdt.
\end{aligned}$$

Combining with (4.10.1), (4.10.2) and (4.10.3), we get

$$-\int_D \varphi_t \eta S(u) dxdt + \int_D S'(u)\eta A(x,t,\nabla u)\nabla \varphi dxdt \le c_5 ||\varphi||_{L^\infty(D)} B.$$

Letting $\delta \to 0$, we get

$$-\int_D \varphi_t \eta T_k(u) dxdt + \int_D \eta A(x,t,\nabla T_k(u))\nabla \varphi dxdt \le c_5 ||\varphi||_{L^\infty(D)} B.$$

By density, we can take $\varphi = T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)$,

$$\begin{aligned}
&-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)\right)\eta T_k(u) dxdt \\
&\qquad + \int_D \eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu) dxdt \le c_5 \varepsilon B.
\end{aligned}$$

Using integration by part, we have

$$\begin{aligned}
&-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\
&\quad=\frac{1}{2}\int_D (T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu))^2\eta_t dxdt\\
&\quad+\int_D T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\langle T_k(w)\rangle_\nu \eta_t dxdt\\
&\quad+\nu\int_D \eta(T_k(w)-\langle T_k(w)\rangle_\nu)T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\
&\quad\geq -\varepsilon(1+k)||\eta_t||_{L^1(D)}+\nu\int_D \eta\left(T_k(w)-\langle T_k(w)\rangle_\nu\right)T_\varepsilon(T_k(u)-\langle T_k(w)\rangle_\nu)dxdt,
\end{aligned}$$

which follows (4.3.15). ■

Proof of the proposition 4.3.17. Let $S_k\in W^{2,\infty}(\mathbb{R})$ such that $S_k(z)=z$ if $|z|\leq k$ and $S_k(z)=sign(z)2k$ if $|z|>2k$. For $m\in\mathbb{N}$, let η_m be the cut off function on D_m with respect to D_{m+1}. It is easy to see that from the assumption and Remark 4.3.4, Proposition 4.3.15 we get $U_{m,n}=\eta_m S_k(v_n)$, $v_n=u_n-h_n$

$$\begin{aligned}
&\sup_{n\geq m+1}\Big(||\left(U_{m,n}\right)_t||_{L^2(-m^2,m^2,H^{-1}(B_m(0)))+L^1(D_m)}+||U_{m,n}||_{L^2(-m^2,m^2,H_0^1(B_m(0)))}\\
&\qquad+||u_n||_{L^1(D_m)}+||v_n||_{L^1(D_m)}\Big)\leq M_m<\infty.
\end{aligned}$$

Thus, $\{U_{m,n}\}_{n\geq m+1}$ is relatively compact in $L^1(D_m)$. On the other hand, for any $n_1,n_2\geq m+1$

$$\begin{aligned}
&|\{|v_{n_1}-v_{n_2}|>\lambda\}\cap D_m|=|\{|\eta_m v_{n_1}-\eta_m v_{n_2}|>\lambda\}\cap D_m|\\
&\quad\leq\frac{1}{k}\left(||v_{n_1}||_{L^1(D_m)}+||v_{n_2}||_{L^1(D_m)}\right)+\frac{1}{\lambda}||\eta_m S_k(v_{n_1})-\eta_m S_k(v_{n_2})||_{L^1(D_m)}\\
&\quad\leq\frac{2M_m}{k}+\frac{1}{\lambda}||U_{m,n_1}-U_{m,n_2}||_{L^1(D_m)},
\end{aligned}$$

and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. So, for any $m\in\mathbb{N}$ there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ is a Cauchy sequence (in measure) in D_m. Therefore, there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ converges to u a.e in $\mathbb{R}^{N+1}$ for some u. Clearly, $u\in L^1_{\text{loc}}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$. Now, we prove that $\nabla u_n\to\nabla u$ a.e in $\mathbb{R}^{N+1}$.

From (4.3.15) with $D=D_{m+2}$, $\eta=\eta_m$ and $T_k(w)=T_k(\eta_{m+1}u)$ we have

$$\begin{aligned}
&\nu\int_{D_{m+2}}\eta_m\left(T_k(\eta_{m+1}u)-\langle T_k(\eta_{m+1}u)\rangle_\nu\right)T_\varepsilon(T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt\\
&\quad+\int_{D_{m+2}}\eta_m A(x,t,\nabla T_k(u_n))\nabla T_\varepsilon(T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt\\
&\quad\leq c_1\varepsilon(1+k)B(n,m)\quad\forall\, n\geq m+2,
\end{aligned}\tag{4.10.5}$$

where

$$B(n,m) = ||(\eta_m)_t(|u_n|+1)||_{L^1(D_{m+2})}$$
$$+\int_{D_{m+2}} (|u_n|+1)^{q_0}\eta dxdt + \int_{D_{m+2}} |\nabla \eta_m^{1/q_1}|^{q_1} dxdt + \int_{D_{m+2}} \eta_m d|\mu_n|,$$

with $q_1 < \frac{q_0-1}{2q_0}$. By the assumption, we verify that the right hand side of (4.10.5) is bounded by $c_2\varepsilon$, where c_2 does not depend on n.
Since $\{\eta_m T_k(u_n)\}_{n\geq m+2}$ is bounded in $L^2(-(m+2)^2,(m+2)^2;H_0^1(B_{m+2}(0)))$, thus there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$\lim_{n\to\infty} \int\limits_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon} \eta_m A(x,t,\nabla T_k(u))\nabla\left(T_k(u_n)-T_k(u)\right)dxdt = 0.$$

Therefore, thanks to $u_n \to u$ a.e in D_{m+2} and $\langle T_k(\eta_{m+1}u)\rangle_\nu \to T_k(\eta_{m+1}u)$ in $L^2(-(m+2)^2,(m+2)^2;H_0^1(B_{m+2}(0)))$, we get

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty} \int\limits_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon} \eta_{1,m}\Phi_{n,k} dxdt \leq c_2\varepsilon \ \forall\ \varepsilon\in(0,1),$$

where $\Phi_{n,k} = \left(A(x,t,T_k(u_n)) - A(x,t,T_k(u))\right)\nabla\left(T_k(u_n)-T_k(u)\right)$. Using Holder inequality,

$$\int_{D_{m+2}} \eta_m\Phi_{k,n}^{1/2} dxdt = \int_{D_{m+2}} \eta_m\Phi_{k,n}^{1/2}\chi_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon} dxdt$$
$$+\int_{D_{m+2}} \eta_m\Phi_{k,n}^{1/2}\chi_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|>\varepsilon} dxdt$$
$$\leq ||\eta_{1,m}||_{L^1(D_{m+2})}^{1/2}\left(\int\limits_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon} \eta_m\Phi_{n,k} dxdt\right)^{1/2}$$
$$+|\{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|>\varepsilon\}\cap D_{m+1}|^{1/2}\left(\int_{D_{m+2}} \eta_m^2\Phi_{k,n} dxdt\right)^{1/2}$$
$$= A_{n,\nu,\varepsilon}.$$

Clearly, $\limsup\limits_{\varepsilon\to0}\limsup\limits_{\nu\to\infty}\limsup\limits_{n\to\infty} A_{n,\nu,\varepsilon} = 0$. It follows

$$\limsup_{n\to\infty}\int_{D_{m+2}} \eta_m\Phi_{k,n}^{1/2} dxdt = 0.$$

Since $\Phi_{n,k}\geq \Lambda_2|\nabla T_k(u_n)-\nabla T_k(u)|^2$, thus $\nabla T_k(u_n)\to\nabla T_k(u)$ in $L^1(D_m)$.
Note that

$$|\{|\nabla u_{n_1}-\nabla u_{n_2}|>\lambda\}\cap D_m| \leq \frac{1}{k}\left(||u_{n_1}||_{L^1(D_m)}+||u_{n_2}||_{L^1(D_m)}\right)$$
$$+\frac{1}{\lambda}|||\nabla T_k(u_{n_1})-\nabla T_k(u_{n_2})||_{L^1(D_m)}$$
$$\leq \frac{2M_m}{k}+\frac{1}{\lambda}|||\nabla T_k(u_{n_1})-\nabla T_k(u_{n_2})|||_{L^1(D_m)}.$$

Thus, we can show that there is a subsequence of $\{\nabla u_n\}$ still denoted by $\{\nabla u_n\}$ converging ∇u a.e in $\mathbb{R}^{N+1}$. $\blacksquare$

Bibliographie

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. **42**, no. 4, 765-778 (1975).

[2] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften **31**, Springer-Verlag (1999).

[3] D.R. Adams, R.J. Bagby, *Translation-dilation invariant estimates for Riesz potentials*, Indiana Univ. Math. J. **23**, 1051-1067 (1974).

[4] R.J. Bagby, *Lebesgue spaces of parabolic potentials*, Ill. J. Math. **15**, 610-634 (1971).

[5] P. Baras and M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[6] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, 185-212 (1985).

[7] P. Baroni, A. D. Castro, G. Palatucci, *Global estimates for nonlinear parabolic equations* To appear in J. Evol. Equations.

[8] D. Blanchard and A. Porretta, *Nonlinear parabolic equations with natural growth terms and measure initial data,* Ann.Scuola Norm. Su. Pisa Cl Sci. **30**, 583-622 (2001).

[9] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Arch. Ration. Mech. Anal. **107**, 293-324 (1989).

[10] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, in : Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. **8**, 23-34 (2002).

[11] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Adv. Nonlinear Stud. **3**, 25-63 (2003).

[12] M. F. Bidaut-Véron, S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84**, 1-49 (2001).

[13] M. F. Bidaut-Véron, Quoc-Hung Nguyen, *Stability properties for quasilinear parabolic equations with measure data.* to appear Journal of European Mathematical Society.

[14] M. F. Bidaut-Véron, Quoc-Hung Nguyen, *Evolution equations of p-Laplace type with absorption or source terms and measure data*, submitted.

[15] M. F. Bidaut-Véron, Quoc-Hung Nguyen, *The porous medium and $p-$Laplacian evolution equations with absorption and measure data*, submitted.

[16] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, Journal des Math. Pures Appl. **102**, 315-337 (2014).

[17] L. Boccardo, A. Dall'Aglio, T. Gallouet and L. Orsina, *Nonlinear parabolic equations with measure data*, J. Funct. Anal. **147**, 237-258 (1997).

[18] Brezis H. and Friedman A., *Nonlinear parabolic equations involving measures as initial conditions,* J.Math.Pures Appl. **62** , 73-97 (1983).

[19] S.S. Byun, J. Ok, S. Ryu, *Global gradient estimates for general nonlinear parabolic equations in nonsmooth domains.* J. Differential Equations, **254** , no. 11, 4290-4326 (2013).

[20] S.S. Byun, L. Wang, *Parabolic equations with BMO nonlinearity in Reifenberg domains*, J. Reine Angew. Math. **615**, 1-24 (2008).

[21] S.S. Byun, L. Wang, *Parabolic equations in time dependent Reifenberg domains*, Adv. Math. **212** , no. 2, 797-818. (2007).

[22] S.S. Byun, L. Wang, *Parabolic equations in Reifenberg domains*, Arch. Ration. Mech. Anal. **176** , no. 2, 271-301 (2005).

[23] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa, **28** , 741-808 (1999).

[24] A. Dall'Aglio and L. Orsina, *Existence results for some nonlinear parabolic equations with nonregular data*, Diff. Int. Equ. **5**, 1335-1354 (1992).

[25] E. DiBenedetto, *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, (1993).

[26] J. Droniou, A. Porretta and A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Anal. **19**, 99-161 (2003).

[27] F. Duzaar, G. Mingione, *Gradient estimate via non-linear potentials*, the American Journal of Mathematics, **133**, no. 4, 1093-1149 (2011).

[28] D. Feyel, A. de la Pradelle, *Topologies fines et compactifications associées à certains espaces de Dirichlet*, Ann. Inst. Fourier Grenoble **27**, 121-146 (1977).

[29] A. Friedman : *Partial differential equations of parabolic type*, Prentice Hall, (1964).

[30] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A Math. pp. **16** 105-113, (1966).

[31] R. Gariery and W. P. Ziemer, *Thermal Capacity and Boundary Regularity* , J. Differential Equations **45**, 374-388 (1982).

[32] D. Gilbarg and N.S. Trudinger, *Elliptic partial dierential equations of second order*, 2nd Springer-Verlag, Berlin-Heidelberg-New York-Tokyo,(1983).

[33] L. Grafakos,*Classical and Modern Fourier Analysis*, Pearson Education, Inc., Upper Saddle River, NJ, (2004), xii+931 pp.

[34] L. I. Hedberg and T. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble), **33**, 161-187 (1983).

[35] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, (2006) xii+404 pp.

[36] P. Honzik, B. Jaye, *On the good-λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180, (2012).

[37] C. Kenig and T. Toro, *Free boundary regularity for harmonic measures and the Poisson kernel*, Ann. Math. **150**, 367-454 (1999).

[38] C. Kenig and T. Toro, *Poisson kernel characterization of Reifenberg at chord arc domains*, Ann. Sci.Ecole Norm. Sup. **36**, 323-401 (2003).

[39] T. Kilpelainen, J. Malý, *Degenerate elliptic equation with measure data and nonlinear potentials*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **19**, 591-613 (1992).

[40] T. Kilpeläinen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).

[41] T. Kilpelainen, P. Koskela, *Global integrability of the gradients of solutions to partial differential equations*. Nonlinear Anal. **23**, no. 7, 899-909 (1994).

[42] T. Kuusi, G. Mingione, *Riesz potentials and nonlinear parabolic equations* J. archive for rational mechanics analysis **212** 727-780 (2014).

[43] T. Kuusi, G. Mingione, *the wolff gradient bound for degenerate parabolic equations* J. Eur. Math. Soc. **16**, 835-892 (2014).

[44] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. UralíCeva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, Amer. Math. Soc., Providence, (1968).

[45] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial boundary-value problems,* Proc. Royal Soc. Edinburg Sect A, **89**, 217-237 (1981).

[46] T. Leonori and F. Petitta, *Local estimates for parabolic equations with nonlinear gradient terms,* Calc. Var. Partial Diff. Equ. **42**, 153-187 (2011).

[47] J.L. Lewis, *Uniformly fat sets*, Trans. Math. Soc. **308**, 177-196 (1988).

[48] G.M. Lieberman, *Boundary regularity for solutions of degenerate parabolic equations*, Nonlinear Anal. **14**, no. 6, 501-524 (1990).

[49] G.M. Lieberman, *Boundary and initial regularity for solutions of degenerate parabolic equations*, Nonlinear Anal. **20**, no. 5, 551-569 (1993).

[50] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific press, River Edge, (1996).

[51] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*. Math. Surveys Monogr. 51, Amer. Math. Soc., Providence, RI, (1997).

[52] V.G.Maz'ya, E.I. Verbitsky, E.I., *Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers*. Ark. Mat. **33**, 81-115 (1995).

[53] T. Mengesha and N. C. Phuc, *Global estimates for quasilinear elliptic equations on Reifenberg flat domains*. Archive for Rational Mechanics and Analysis **203**, 189-216 (2012).

[54] T. Mengesha and N. C. Phuc, *Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains*. Journal of Differential Equations **250**, 1485-2507 (2011).

[55] Mingione, Giuseppe Gradient estimates below the duality exponent. Math. Ann. **346**, no. 3, 571âĂŞ627 (2010).

[56] G. Mingione : *Nonlinear measure data problems* Milan journal of mathematics **79**, 429-496 (2011).

[57] P. Mikkonen, *On the Wolff potential and quasilinear elliptic equations involving measures*, Ann. Acad. Sci. Fenn., Ser AI, Math. Dissert. **104**, 1-71 (1996).

[58] J. Moser, *A Harnack inequality for parabolic differential equations.* Comm. Pure Appl. Math. **17**, 101-134 (1964). Corrections in : Comm. Pure Appl. Math. **20**, 231-236 (1967).

[59] B. Muckenhoupt and R. Wheeden, *Weight norm inequality for fractional integrals* Trans. A.M.S, **192**, 279-294 (1974).

[60] J. Naumann and J. Wolf, *Interior integral estimates on weak solutions of nonlinear parabolic systems.* Inst. fur Math., Humboldt Universitet, Bonn (1994).

[61] Phuoc-Tai Nguyen, *Parabolic equations with exponential nonlinearity and measure data* arXiv :1312.2509.

[62] Quoc-Hung Nguyen and L. Véron, *Quasilinear and Hessian type equations with exponential reaction and measure data*, Archive for Rational Mechanics and Analysis, **214**, 235-267 (2014).

[63] Quoc-Hung Nguyen and L. Véron, *Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain* submitted.

[64] Quoc-Hung Nguyen, *Mild Solutions of the Navier-Stokes Equations*, works in progress.

[65] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data,* Ann. Math. Pura Appl. **187**, 563-604 (2008).

[66] F. Petitta, A. Ponce and A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ. **11**, 861-905 (2011).

[67] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations,* Ann. Mat. Pura Apll. **177**, 143-172 (1999).

[68] N. C. Phuc, I. E. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type*, Ann. Math. 168, 859-914 (2008).

[69] N. C. Phuc, I. E. Verbitsky, *Singular quasilinear and Hessian equation and inequalities*, J. Functional Analysis,**256**, 1875-1906 (2009).

[70] N.C.Phuc, *Global integral gradient bounds for quasilinear equations below or near the natural exponent.* To appear in Arkiv for Matematik.

[71] N. C. Phuc, *Nonlinear Muckenhoupt-Wheeden type bounds on Reifenberg flat domains*, with applications to quasilinear Riccati type equations. To appear in Advances in Mathematics.

[72] N. C. Phuc, *Morrey global bounds and quasilinear Riccati type equations below the natural exponent.* To appear in Journal des Mathématiques Pures et Appliquées.

[73] P. Quittner, P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhauser Advanced Texts, (2007), 584 p.+xi. ISBN : 978-3-7643-8441-8

[74] E. Reinfenberg, *Solutions of the Plateau Problem for m-dimensional surfaces of varying topological type*, Acta Math, **104**, 1-92 (1960).

[75] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailiov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin and New York, (1995).

[76] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, **30**, Princeton University Press, Princeton, (1970).

[77] E.M. Stein, *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integral* **43**, Princeton University Press, Princeton, (1993).

[78] T. Toro, *Doubling and flatness : geometry of measures*, Notices Amer. Math. Soc. **44**, 1087-1094 (1997).

[79] B. O. Tureson, *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, **1736**, Springer-Verlag (2000).

[80] I.E. Verbitsky, *Nonlinear potential and trace inequalities* Oper. Theory, Adv. Appl. **110**, 323-343 (1999).

[81] I.E. Verbitsky and Richard L. Wheeden, *Weighted norm inequalities for integral operators*. Transactions of the A.M.S **350**, 8, 3371-3391, (1988).

[82] L. Véron, *Elliptic equations involving measures*, in Stationary Partial Differential Equations, vol. I, Handbook of Equations, Elsevier B.V., pp. 593-712 (2004).

[83] W.P. Ziemer, *Behavior at the Boundary of Solutions of Quasilinear Parabolic Equations*, J. D.E, **35**, 291-305 (1980).

Chapitre 5

Pointwise estimates and existence of solutions of porous medium and p-Laplace evolution equations with absorption and measure data

Abstract

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$. We obtain a necessary and a sufficient condition, expressed in terms of capacities, for existence of a solution to the porous medium equation with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases}$$

where σ and μ are bounded Radon measures, $q > \max(m, 1)$, $m > \frac{N-2}{N}$. We also obtain a sufficient condition for existence of a solution to the p-Laplace evolution equation

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma. \end{cases}$$

where $q > p - 1$ and $p > 2$.

5.1 Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$, and $\Omega_T = \Omega \times (0,T)$. In this paper we study the existence of solutions to the following two types of evolution problems : the porous medium problem with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases} \tag{5.1.1}$$

where $m > \frac{N-2}{N}$ and $q > \max(1,m)$, and the p-Laplace evolution problem with absorption

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases} \tag{5.1.2}$$

where $q > p - 1 > 1$, and μ and σ are bounded Radon measures respectively on Ω_T and Ω. In the sequel, for any bounded domain O of $\mathbb{R}^l (l \geq 1)$, we denote by $\mathfrak{M}_b(O)$ the set of bounded Radon measures in O, and by $\mathfrak{M}_b^+(O)$ its positive cone. For any $\nu \in \mathfrak{M}_b(O)$, we denote by ν^+ and ν^- respectively its positive and negative part.

When $m = 1, p = 2$ and $q > 1$ the problem has been studied by Brezis and Friedman [13] with $\mu = 0$. It is shown that in the subcritical case $q < 1 + 2/N$, the problem can be solved for any $\sigma \in \mathfrak{M}_b(\Omega)$, and it has no solution when $q \geq 1 + 2/N$ and σ is a Dirac mass. The general case has been solved by Baras and Pierre [5] and their results are expressed in terms of capacities. For $s > 1, \alpha > 0$, the capacity $\text{Cap}_{\mathbf{G}_\alpha,s}$ of a Borel set $E \subset \mathbb{R}^N$, defined by

$$\text{Cap}_{\mathbf{G}_\alpha,s}(E) = \inf\{||g||^s_{L^s(\mathbb{R}^N)} : g \in L^s_+(\mathbb{R}^N), \mathbf{G}_\alpha * g \geq 1 \text{ on } E\},$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α and the capacity $\text{Cap}_{2,1,s}$ of a compact set $K \subset \mathbb{R}^{N+1}$ is defined by

$$\text{Cap}_{2,1,s}(K) = \inf\left\{||\varphi||^s_{W_s^{2,1}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\right\},$$

where

$$||\varphi||_{W_s^{2,1}(\mathbb{R}^{N+1})} = ||\varphi||_{L^s(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^s(\mathbb{R}^{N+1})} + ||\,|\nabla\varphi|\,||_{L^s(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\ldots,N} ||\varphi_{x_i x_j}||_{L^s(\mathbb{R}^{N+1})}.$$

The capacity $\text{Cap}_{2,1,s}$ is extended to Borel sets by the usual method. Note the relation between the two capacities :

$$C^{-1}\text{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E) \leq \text{Cap}_{2,1,s}(E \times \{0\}) \leq C\text{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E)$$

for any Borel set $E \subset \mathbb{R}^N$, see [35, Corollary 4.21]. In particular, for any $\omega \in \mathfrak{M}_b(\mathbb{R}^N)$ and $a \in \mathbb{R}$, the measure $\omega \otimes \delta_{\{t=a\}}$ in $\mathbb{R}^{N+1}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,s}$ (in $\mathbb{R}^{N+1}$) if and only if ω is absolutely continuous with respect to the

capacity $\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}$ (in $\mathbb{R}^N$).
From [5], the problem

$$\begin{cases} u_t - \Delta u + |u|^{q-1}u = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases}$$

has a solution if and only if the measures μ and σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω respectively, where $q' = \frac{q}{q-1}$.

In Section 5.2 we study problem (5.1.1).

For $m > 1$, Chasseigne [15] has extended the results of [13] for $\mu = 0$ in the new subcritical range $m < q < m + \frac{2}{N}$. The supercritical case $q \geq m + \frac{2}{N}$ with $\mu = 0$ and σ is positive is studied in [14]. He has essentially proved that if problem (5.1.1) has a solution, then $\sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}$, defined for any compact set $K \subset \mathbb{R}^{N+1}$ by

$$\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = \inf\left\{ ||\varphi||^{\frac{q}{q-m}}_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^N), \varphi \geq 1 \text{ in a neighborhood of } E \right\},$$

where

$$||\varphi||_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} = ||\varphi||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^{q'}(\mathbb{R}^{N+1})} + ||\,|\nabla\varphi|\,||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} ||\varphi_{x_i x_j}||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})}.$$

In this Section, we first give *necessary conditions* on the measures μ and σ for existence, which cover the results mentioned above.

Theorem 5.1.1 *Let $q > \max(1,m)$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. If problem (5.1.1) has a very weak solution then μ and $\sigma \otimes \delta_{\{t=0\}}$ are absolutely continuous with respect to the capacity $Cap_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$.*

Remark 5.1.2 *It is easy to see that the capacity $Cap_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$ is absolutely continuous with respect to the capacity $Cap_{2,1,\frac{q}{q-\max\{m,1\}}}$. Therefore μ and $\sigma\otimes\delta_{\{t=0\}}$ are absolutely continuous with respect to the capacities $Cap_{2,1,\frac{q}{q-\max\{m,1\}}}$. In particular σ is absolutely continuous with respect to the capacity $Cap_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$.*

The main result of this Section is the following *sufficient condition* for existence, where we use the notion of R-truncated Riesz parabolic potential $\mathbb{I}_2$ on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathfrak{M}^+_b(\Omega_T)$, defined by

$$\mathbb{I}_2^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \quad \text{for any} \quad (x,t) \in \mathbb{R}^{N+1},$$

with $R \in (0,\infty]$, and $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t+\rho^2)$.

Theorem 5.1.3 *Let $m > \frac{N-2}{N}$, $q > \max(1, m)$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$.*

i. *If $m > 1$ and μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω, then there exists a very weak solution u of (5.1.1), satisfying for a.e in Ω_T*

$$|u| \le C\left(\left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\sigma|(\Omega) + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|]\right), \tag{5.1.3}$$

where $C = C(N, m) > 0$ and

$$m_1 = \frac{(N+2)(2mN+1)}{m(mN+2)(1+2N)}, \qquad d = diam(\Omega) + T^{1/2}.$$

ii. *If $\frac{N-2}{N} < m \le 1$, and μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,\frac{2q}{2(q-1)+N(1-m)}}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ in Ω, there exists a very weak solution u of (5.1.1), such that for a.e in Ω_T*

$$|u| \le C\left(\left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d}[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|]\right)^{\frac{2}{2-N(1-m)}}\right), \tag{5.1.4}$$

where $C = C(N, m) > 0$ and

$$m_2 = \frac{2N(N+2)(m+1)}{(2+Nm)(2-N(1-m))(2+N(1+m))}.$$

.

Remark 5.1.4 *These estimates are not homogeneous in u. In particular if $\mu \equiv 0$, u satisfies the decay estimates, for a.e. $(x,t) \in \Omega_T$,*

i. *if $m > 1$,*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)}{d^N}\right)^{m_1} + |\sigma|(\Omega) + 1 + \frac{|\sigma|(\Omega)}{Nt^{N/2}}\right),$$

ii. *if $m < 1$,*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)}{d^N}\right)^{m_2} + 1 + \left(\frac{|\sigma|(\Omega)}{Nt^{N/2}}\right)^{\frac{2}{2-N(m-1)}}\right).$$

We also give other types of *sufficient* conditions for *measures which are good in time*, that means such that

$$\sigma \in L^1(\Omega) \quad \text{and } |\mu| \le f + \omega \otimes F, \quad \text{where } f \in L^1_+(\Omega_T), F \in L^1_+((0,T)), \tag{5.1.5}$$

see Theorem 5.2.10. The proof is based on estimates for the stationary problem in terms of elliptic Riesz potential.

In Section 5.3, we consider problem (5.1.2). Let us recall some former results about it.

For $q > p - 1 > 0$, Pettitta, Ponce and Porretta [37] have proved that it admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ is a *diffuse* measure, i.e. absolutely continuous with respect to C_p-capacity in Ω_T, defined on a compact set $K \subset \Omega_T$ by

$$C_p(K, \Omega_T) = \inf \left\{ ||\varphi||_W : \varphi \in C_c^\infty(\Omega_T),\ \varphi \geq 1 \text{ on } K \right\}, \tag{5.1.6}$$

where

$$W = \{z : z \in L^p(0, T, W_0^{1,p}(\Omega) \cap L^2(\Omega)), z_t \in L^{p'}(0, T, W^{-1,p'}(\Omega) + L^2(\Omega))\}.$$

embedded with the norm

$$||z||_W = ||z||_{L^p((0,T);W_0^{1,p}(\Omega) \cap L^2(\Omega))} + ||z_t||_{L^{p'}((0,T);W^{-1,p'}(\Omega)+L^2(\Omega))}.$$

In the recent work [7, 8], we have proved a stability result for the p-Laplace parabolic equation, see Theorem 5.3.5, for $p > \frac{2N+1}{N+1}$. As a first consequence, in the new subcritical range

$$q < p - 1 + \frac{p}{N},$$

problem (5.1.2) admits a renormalized solution for any measures $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in L^1(\Omega)$. Moreover, we have obtained sufficient conditions for existence, for measures that have a *good behavior in time*, of the form (5.1.5). It is shown that (5.1.2) has a renormalized solution if $\omega \in \mathfrak{M}_b^+(\Omega)$ is absolutely continuous with respect to $\text{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}$. The proof is based on estimates of [9] for the stationary problem which involve Wolff potentials.

Here we give *new sufficient conditions when* $p > 2$. The next Theorem is our second main result :

Theorem 5.1.5 *Let $q > p - 1 > 1$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. If μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω, then there exists a distribution solution of problem (5.1.2) which satisfies the pointwise estimate*

$$|u| \leq C \left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N} \right)^{m_3} + \mathbb{I}_2^{2D} \left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu| \right] \right) \tag{5.1.7}$$

for a.e in Ω_T with $C = C(N, p)$ and

$$m_3 = \frac{(N+p)(\lambda+1)(p-1)}{((p-1)N+p)(1+\lambda(p-1))}, \quad \lambda = \min\{1/(p-1), 1/N\}, \quad D = diam(\Omega) + T^{1/p}. \tag{5.1.8}$$

Moreover, if $\sigma \in L^1(\Omega)$, u is a renormalized solution.

5.2 Porous medium equation

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. The solutions of (5.1.1) are considered in a weak sense :

Definition 5.2.1 *Let $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$ and $g \in C(\mathbb{R})$.*
i. *A function u is a weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + g(u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad in\ \ \Omega. \end{cases} \tag{5.2.1}$$

if $u \in C([0,T]\,; L^2(\Omega))$, $|u|^m \in L^2((0,T); H_0^1(\Omega))$ and $g(u) \in L^1(\Omega_T)$, and for any $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,

$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} \nabla(|u|^{m-1}u).\nabla\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0) d\sigma.$$

ii. *A function u is a very weak solution of (5.2.1) if $u \in L^{\max\{m,1\}}(\Omega_T)$ and $g(u) \in L^1(\Omega_T)$, and for any $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,*

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0) d\sigma.$$

First we give a priori estimates for the problem without perturbation term :

Proposition 5.2.2 *Let $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2((0,T); H_0^1(\Omega))$ be a weak solution to problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad in\ \ \Omega, \end{cases} \tag{5.2.2}$$

with $\sigma \in C_b(\Omega)$ and $\mu \in C_b(\Omega_T)$. Then,

$$||u||_{L^\infty((0,T);L^1(\Omega))} \leq |\sigma|(\Omega) + |\mu|(\Omega_T), \tag{5.2.3}$$

$$||u||_{L^{m+2/N,\infty}(\Omega_T)} \leq C_1(|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{mN+2}}, \tag{5.2.4}$$

$$|||\nabla(|u|^{m-1}u)|||_{L^{\frac{mN+2}{mN+1},\infty}(\Omega_T)} \leq C_2(|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{m(N+1)+1}{mN+2}}, \tag{5.2.5}$$

where $C_1 = C_1(N,m), C_2 = C_2(N,m)$.

Proof of Proposition 5.2.2. For any $\tau \in (0,T)$, and $k > 0$ we have

$$\int_{\Omega_\tau} (H_k(u))_t dxdt + \int_{\Omega_\tau} |\nabla T_k(|u|^{m-1}u)|^2 dxdt = \int_{\Omega_\tau} T_k(|u|^{m-1}u) d\mu(x,t),$$

where $H(a) = \int_0^a T_k(|y|^{m-1}y)dy$. This leads to

$$\int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt \leq k(|\sigma|(\Omega) + |\mu|(\Omega_T)) \quad \text{and} \tag{5.2.6}$$
$$\int_{\Omega} (H_k(u))(\tau) dx \leq k(|\sigma|(\Omega) + |\mu|(\Omega_T)),\ \forall \tau \in (0,T).$$

Since $H_k(a) \geq k(|a| - k^{\frac{1}{m}})$ for any a and $k > 0$, we find

$$\int_\Omega (|u|(\tau) - k^{\frac{1}{m}})dx \leq |\sigma|(\Omega) + |\mu|(\Omega_T),\ \forall \tau \in (0,T).$$

Letting $k \to 0$, we get (5.2.3).
Next we prove (5.2.4). By the Gagliardo-Nirenberg embedding theorem, there holds

$$\begin{aligned}\int_{\Omega_T} |T_k(|u|^{m-1}u)|^{\frac{2(N+1)}{N}} dxdt &\leq C_1 ||T_k(|u|^{m-1}u)||^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt \\ &\leq C_1 k^{\frac{2(m-1)}{mN}} ||u||^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt.\end{aligned}$$

Thus, from (5.2.6) and (5.2.3) we get

$$k^{\frac{2(N+1)}{N}} |\{|u|^m > k\}| \leq \int_{\Omega_T} |T_k(|u|^{m-1}u)|^{\frac{2(N+1)}{N}} dxdt \leq c_1 k^{\frac{2(m-1)}{mN}+1} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{N}},$$

which implies (5.2.4). Finally, we prove (5.2.5). Thanks to (5.2.6) and (5.2.4) we have for $k, k_0 > 0$

$$\begin{aligned}|\{|\nabla(|u|^{m-1}u)| > k\}| &\leq \frac{1}{k^2}\int_0^{k^2} |\{|\nabla(|u|^{m-1}u)| > \ell\}| d\ell \\ &\leq |\{|u|^m > k_0\}| + \frac{1}{k^2}\int_{\Omega_T} |\nabla T_{k_0}(|u|^{m-1}u)|^2 dxdt \\ &\leq C_1 k_0^{-\frac{2}{mN}-1} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{N}} + k_0 k^{-2} (|\sigma|(\Omega) + |\mu|(\Omega_T)).\end{aligned}$$

Choosing $k_0 = k^{\frac{Nm}{Nm+1}} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{m}{Nm+1}}$, we get (5.2.5). ■

Next we show the necessary conditions given at Theorem 5.1.1.
Proof of Theorem 5.1.1. As in [5, Proof of Proposition 3.1], it is enough to claim that for any compact $K \subset \Omega \times [0,T)$ such that $\mu^-(K) = 0$, $(\sigma^- \otimes \delta_{\{t=0\}})(K) = 0$ and $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = 0$ then $\mu^+(K) = 0$ and $(\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. Let $\varepsilon > 0$ and choose an open set O such that $(|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(O \backslash K) < \varepsilon$ and $K \subset O \subset \Omega \times (-T,T)$. One can find a sequence $\{\varphi_n\} \subset C_c^\infty(O)$ which satisfies $0 \leq \varphi_n \leq 1$, $\varphi_n|_K = 1$ and $\varphi_n \to 0$ in $W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})$ and almost everywhere in O (see [5, Proposition 2.2]). We get

$$\begin{aligned}\int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma &= -\int_{\Omega_T} u(\varphi_n)_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi_n dxdt + \int_{\Omega_T} |u|^{q-1}u\varphi_n dxdt \\ &\leq (||u||_{L^q(\Omega_T)} + ||u||^m_{L^q(\Omega_T)}) ||\varphi_n||_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})} + \int_{\Omega_T} |u|^q \varphi_n dxdt.\end{aligned}$$

Note that

$$\begin{aligned}\int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma &\geq \mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) - (|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(O \backslash K) \\ &\geq \mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) - \varepsilon.\end{aligned}$$

This implies

$$\mu^+(K)+(\sigma^+\otimes\delta_{\{t=0\}})(K) \le (||u||_{L^q(\Omega_T)}+||u||^m_{L^q(\Omega_T)})||\varphi_n||_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})}+\int_{\Omega_T}|u|^q\varphi_n dxdt+\varepsilon.$$

Letting the limit we get $\mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) \le \varepsilon$. Therefore, $\mu^+(K) = (\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. ∎

Next we look for sufficient conditions of existence . The crucial result used to establish Theorem 5.1.3 is the following a priori estimates, due to of Liskevich and Skrypnik [32] for $m \ge 1$ and Bogelein, Duzaar and Gianazza [12] for $m \le 1$.

Theorem 5.2.3 *Let $m > \frac{N-2}{N}$ and $\mu \in (C_b(\Omega_T))^+$. Let $u \in L^\infty_+(\Omega_T)$ with $u^m \in L^2(0,T,H^1_{loc}(\Omega))$ be a weak solution to equation*

$$u_t - \Delta(u^m) = \mu \quad in\ \Omega_T.$$

Then there exists $C = C(N,m)$ such that, for almost all $(y,\tau) \in \Omega_T$ and any cylinder $\tilde{Q}_r(y,\tau) = B_r(y) \times (\tau - r^2, \tau + r^2) \subset\subset \Omega_T$, there holds

i. *if $m > 1$*

$$u(y,\tau) \le C\left(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,\tau)}|u|^{m+\frac{1}{2N}}dxdt\right)^{\frac{2N}{1+2N}} + ||u||_{L^\infty((\tau-r^2,\tau+r^2);L^1(B_r(y)))} + 1\right)$$
$$+ C\mathbb{I}_2^{2r}[\mu](y,\tau),$$

ii. *if $m \le 1$,*

$$u(y,\tau) \le C\left(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,s)}|u|^{\frac{2(1+mN)}{N(1+m)}}dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1\right)$$
$$+ C\left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}}$$

As a consequence we get a new priori estimate for the porous medium equation :

Corollary 5.2.4 *Let $m > \frac{N-2}{N}$ and $\mu \in C_b(\Omega_T)$. Let $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2(0,T,H^1_0(\Omega))$ be the weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu & in\ \Omega_T,\\ u = 0 & on\ \partial\Omega \times (0,T),\\ u(0) = 0 & in\ \Omega. \end{cases}$$

Then there exists $C = C(N,m)$ such that, for a.e. $(y,\tau) \in \Omega_T$,

i. *if $m > 1$,*

$$|u(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right), \tag{5.2.7}$$

ii. *if* $m \leq 1$,

$$|u(y,\tau)| \leq C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right), \tag{5.2.8}$$

where m_1, m_2 *and* d *are defined in Theorem 5.1.3.*

Proof. Let $x_0 \in \Omega$, and $Q = B_{2d}(x_0) \times (-(2d)^2, (2d)^2)$. Consider the function $U \in (C_b(Q))^+$, with $U^m \in L^p((-(2d)^2, (2d)^2); H_0^1(B_{2d}(x_0)))$ such that U is weak solution of

$$\begin{cases} U_t - \Delta(U^m) = \chi_{\Omega_T}|\mu| & \text{in } B_{2d}(x_0) \times (-(2d)^2, (2d)^2), \\ U = 0 & \text{on } \partial B_{2d}(x_0) \times (-(2d)^2, (2d)^2), \\ U(-(2d)^2) = 0 & \text{in } B_{2d}(x_0). \end{cases} \tag{5.2.9}$$

From Theorem 5.2.3, we get, for a.e $(y,\tau) \in \Omega_T$,

$$U(y,\tau) \leq c_1\left(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,\tau)} |U|^{m+\frac{1}{2N}}dxdt\right)^{\frac{2N}{1+2N}} + ||U||_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} + 1\right)$$
$$+ c_1\mathbb{I}_2^{2d}[|\mu|](y,\tau),$$

if $m > 1$ and

$$U(y,\tau) \leq c_1\left(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,s)} |u|^{\frac{2(1+mN)}{N(1+m)}}dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1\right)$$
$$+ c_1\left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}},$$

if $m \leq 1$. By Proposition 5.2.2, we have

$$||U||_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} \leq |\mu|(\Omega_T),$$
$$|\{|U| > \ell\}| \leq c_2(|\mu|(\Omega_T))^{\frac{2+N}{N}}\ell^{-\frac{2}{N}-m} \quad \forall \ell > 0.$$

Thus, for any $\ell_0 > 0$,

$$\int_Q U^{m+\frac{1}{2N}}dxdt = (m+\frac{1}{2N})\int_0^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell$$
$$= (m+\frac{1}{2N})\int_0^{\ell_0} \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell + (m+\frac{1}{2N})\int_{\ell_0}^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell$$
$$\leq c_3 d^{N+2}\ell_0^{m+\frac{1}{2N}} + c_4\ell_0^{\frac{1}{2N}-\frac{2}{N}}(|\mu|(\Omega_T))^{\frac{2+N}{N}}.$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{N+2}{mN+2}}$, we get

$$\int_Q U^{(\lambda+1)(p-1)}dxdt \leq c_5 d^{N+2}\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{(N+2)(2mN+1)}{2mN(mN+2)}}.$$

Thus, for a.e $(y,\tau)\in\Omega_T$,

$$U(y,\tau)\le c_6\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1}+|\mu|(\Omega_T)+1+\mathbb{I}_2^{2d}[|\mu|](y,\tau)\right),$$

if $m>1$. Similarly, we also obtain for a.e $(y,\tau)\in\Omega_T$,

$$U(y,\tau)\le c_7\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2}+1+\left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right),$$

if $m\le 1$. By the comparison principle we get $|u|\le U$ in Ω_T, and (5.2.7)-(5.2.8) follow. ∎

Lemma 5.2.5 *Let $g\in C_b(\mathbb{R})$ be nondecreasing with $g(0)=0$, and $\mu\in C_b(\Omega_T)$. There exists a weak solution $u\in L^\infty(\Omega_T)$ with $|u|^m\in L^2(0,T,H_0^1(\Omega))$ of problem*

$$\left\{\begin{array}{l} u_t-\Delta(|u|^{m-1}u)+g(u)=\mu \quad in\ \Omega_T,\\ u=0 \quad on\ \partial\Omega\times(0,T),\\ u(0)=0 \quad in\ \Omega. \end{array}\right. \tag{5.2.10}$$

Moreover, the comparison principle holds for these solutions : if u_1,u_2 are weak solutions of (5.2.10) when (μ,g) is replaced by (μ_1,g_1) and (μ_2,g_2), where $\mu_1,\mu_2\in C_b(\Omega_T)$ with $\mu_1\ge\mu_2$ and g_1,g_2 have the same properties as g with $g_1\le g_2$ in $\mathbb{R}$ then $u_1\ge u_2$ in Ω_T.

Proof of Lemma 5.2.5. Set $a_n(s)=m|s|^{m-1}$ if $1/n\le|s|\le n$ and $a_n(s)=mn^{m-1}$ if $|s|\ge n$, $a_n(s)=m(1/n)^{m-1}$ if $|s|\le 1/n$. Also $A_n(\tau)=\int_0^\tau a_n(s)ds$. Then one can find u_n being a weak solution to the following equation

$$\left\{\begin{array}{l} (u_n)_t-\mathrm{div}(a_n(u_n)\nabla u_n)+g(u_n)=\mu \quad \text{in } \Omega_T,\\ u_n=0 \quad \text{on } \partial\Omega\times(0,T),\\ u_n(0)=0 \quad \text{in } \Omega. \end{array}\right. \tag{5.2.11}$$

It is easy to see that $|u_n(x,t)|\le t||\mu||_{L^\infty(\Omega_T)}$ for all $(x,t)\in\Omega_T$. Thus, choosing $A_n(u_n)$ as a test function, we obtain

$$\int_{\Omega_T}|\nabla A_n(u_n)|^2dxdt\le C_1(T,||\mu||_{L^\infty(\Omega_T)}). \tag{5.2.12}$$

Now set $\Phi_n(\tau)=\int_0^\tau|A_n(s)|ds$. Choosing $|A_n(u_n)|\varphi$ as a test function in (5.2.11), where $\varphi\in C_c^{2,1}(\Omega_T)$, we get the relation in $\mathcal{D}'(\Omega_T)$:

$$(\Phi_n(u_n))_t-\mathrm{div}(|A_n(u_n)|\nabla A_n(u_n))+\nabla A_n(u_n).\nabla|A_n(u_n)|+|A_n(u_n)|g(u_n)=|A_n(u_n)|\mu.$$

Hence,

$$\begin{aligned} ||(\Phi_n(u_n))_t||_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))}\le{}& ||A_n(u_n)\nabla A_n(u_n)||_{L^2(\Omega_T)}+||\nabla A_n(u_n)|||^2_{L^2(\Omega_T)}\\ &+||A_n(u_n)g(u_n)||_{L^1(\Omega_T)}+||A_n(u_n)\mu||_{L^1(\Omega_T)}. \end{aligned}$$

Combining this with (5.2.12) and the estimate $|A_n(u_n)|\le C_2(T,||\mu||_{L^\infty(\Omega)})$, we deduce that

$$\sup_n||(\Phi_n(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))}<\infty.$$

On the other hand, since $|A_n(u_n)| \leq |u_n|a_n(u_n) \leq T||\mu||_{L^\infty(\Omega)}a_n(u_n)$, there holds

$$\int_{\Omega_T} |\nabla\Phi_n(u_n)|^2 dxdt = \int_{\Omega_T} |A_n(u_n)|^2|\nabla u_n|^2 dxdt \leq T^2||\mu||^2_{L^\infty(\Omega)} \int_{\Omega_T} |a_n(u_n)|^2|\nabla u_n|^2 dxdt$$
$$\leq T^2||\mu||^2_{L^\infty(\Omega)} \int_{\Omega_T} |\nabla A_n(u_n)|^2 dxdt \leq C_3(T, ||\mu||_{L^\infty(\Omega)}).$$

Therefore, $\Phi_n(u_n)$ is relatively compact in $L^1(\Omega_T)$. Note that

$$\Phi_n(s) = \begin{cases} \frac{m}{2}\left(\frac{1}{n}\right)^m |s|^2 \mathrm{sign}(s) & \text{if } |s| \leq \frac{1}{n} \\ (m-1)\left(\frac{1}{n}\right)^m \left(|s| - \frac{1}{n}\right)\mathrm{sign}(s) + \frac{1}{m+1}\left(|s|^{m+1} - \left(\frac{1}{n}\right)^{m+1}\right)\mathrm{sign}(s) & \text{if } \frac{1}{n} \leq |s| \leq n. \end{cases}$$

So, for every $n_1, n_2 \geq n$ and $|s_1|, |s_2| \leq T||\mu||_{L^\infty(\Omega)}$,

$$\frac{1}{m+1}||s_1|^m s_1 - |s_2|^m s_2| \leq C_4(m, T||\mu||_{L^\infty(\Omega)})\left(\frac{1}{n}\right)^m + |\Phi_{n_1}(s_1) - \Phi_{n_2}(s_2)|.$$

Hence, for any $\varepsilon > 0$,

$$\left|\left\{\frac{1}{m+1}||u_{n_1}|^m u_{n_1} - |u_{n_2}|^m u_{n_2}| > 2\varepsilon\right\}\right| \leq |\left\{|\Phi_{n_1}(u_{n_1}) - \Phi_{n_2}(u_{n_2})| > \varepsilon\right\}|,$$

for all $n_1, n_2 \geq \left(C_4(m, T||\mu||_{L^\infty(\Omega)})/\varepsilon\right)^{1/m}$. Thus, up to a subsequence $\{u_n\}$ converges a.e in Ω_T to a function u. From (5.2.11) we can write

$$-\int_{\Omega_T} u_n\varphi_t dxdt - \int_{\Omega_T} A_n(u_n)\Delta\varphi dxdt + \int_{\Omega_T} g(u_n)\varphi dxdt = \int_{\Omega_T} \varphi d\mu,$$

for any $\varphi \in C_c^{2,1}(\Omega_T)$. Thanks to the dominated convergence Theorem we deduce that

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu.$$

By Fatou's lemma and (5.2.12) we also get $|u|^m \in L^2((0,T); H_0^1(\Omega))$.
Furthermore, by the classic maximum principle, see [30, Theorem 9.7], if $\{\tilde{u}_n\}$ is a sequence of solutions to equations (5.2.11) where (g, μ) is replaced by (h, ν) such that $\nu \in C_b(\Omega_T)$ with $\nu \geq \mu$ and h has the same properties as g satisfying $h \leq g$ in $\mathbb{R}$, then, $u_n \leq \tilde{u}_n$. As $n \to \infty$, we get $u \leq \tilde{u}$. This achieves the proof. ∎

Lemma 5.2.6 *Let $m > \frac{N-2}{N}$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, such that $g \in C_b(\mathbb{R})$, $g(0) = 0$, and let $\mu \in \mathfrak{M}_b(\Omega_T)$. There exists a very weak solution u of equation* (5.2.10) *which satisfies* (5.2.7)-(5.2.8) *and*

$$\int_{\Omega_T} |g(u)| dxdt \leq |\mu|(\Omega_T), \quad ||u||_{L^{m+2/N,\infty}(\Omega_T)} \leq C(|\mu|(\Omega_T))^{\frac{N+2}{mN+2}}. \tag{5.2.13}$$

where $C = C(m, N) > 0$. Moreover, the comparison principle holds for these solutions : if u_1, u_2 are very weak solutions of (5.2.10) *when (μ, g) is replaced by (μ_1, g_1) and (μ_2, g_2), where $\mu_1, \mu_2 \in \mathfrak{M}_b(\Omega_T)$ with $\mu_1 \geq \mu_2$ and g_1, g_2 have the same properties as g with $g_1 \leq g_2$ in $\mathbb{R}$ then $u_1 \geq u_2$ in Ω_T.*

Proof. Let $\{\mu_n\}$ be a sequence in $C_c^\infty(\Omega_T)$ converging to μ in $\mathfrak{M}_b(\Omega_T)$, such that $|\mu_n| \leq \varphi_n * |\mu|$ and $|\mu_n|(\Omega_T) \leq |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$ where $\{\varphi_n\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$. By Lemma 5.2.5 and corollary 5.2.4 there exists a very weak solution u_n of problem

$$\begin{cases} (u_n)_t - \Delta(|u_n|^{m-1}u_n) + g(u_n) = \mu_n \text{ in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = 0 \quad \text{ in } \Omega, \end{cases}$$

which satisfies for a.e $(y,\tau) \in \Omega_T$,

$$|u_n(y,\tau)| \leq c_1 \left(\left(\frac{|\mu|(\Omega_T)}{d^N} \right)^{m_1} + |\mu|(\Omega_T) + 1 + \varphi_n * \mathbb{I}_2^{2d}[|\mu|](y,\tau) \right) \quad \text{ if } m > 1,$$
$$|u_n(y,\tau)| \leq c_1 \left(\left(\frac{|\mu|(\Omega_T)}{d^N} \right)^{m_2} + 1 + \left(\varphi_n * \mathbb{I}_2^{2d_1}[|\mu|](y,\tau) \right)^{\frac{2}{2-N(1-m)}} \right) \quad \text{ if } m \leq 1,$$

and

$$\int_{\Omega_T} |g(u_n)| dxdt \leq |\mu|(\Omega_T).$$

Furthermore, by (5.2.4) in Proposition 5.2.2 and (5.2.6) in the proof of Proposition 5.2.2.

$$\int_{\Omega_T} |\nabla T_k(|u_n|^{m-1}u_n)|^2 dxdt \leq k|\mu|(\Omega_T), \qquad \forall k > 0, \tag{5.2.14}$$
$$|\{|u_n| > \ell\}| \leq c_2 \ell^{-\frac{2}{N}-m} |\mu|(\Omega_T)^{\frac{N+2}{N}}, \qquad \forall \ell > 0, \tag{5.2.15}$$

For $l > 0$, we consider $S_l \in C_c^2(\mathbb{R})$ such that

$$S_l(a) = |a|^m a, \quad \text{for } |a| \leq l, \quad \text{and} \quad S_l(a) = (2l)^{m+1}\text{sign}(a), \quad \text{for } |a| \geq 2l.$$

Then we find the relation in $\mathcal{D}'(\Omega_T)$:

$$(S_l(u_n))_t - \text{div}\left(S_l'(u_n)\nabla(|u_n|^{m-1}u_n)\right) + m|u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n) + g(u_n)S_l'(u_n) = S_l'(u_n)\mu_n.$$

It leads to

$$||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))} \leq ||S_l'(u_n)\nabla(|u_n|^{m-1}u_n)||_{L^2(\Omega_T)}$$
$$+ m|||u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n)||_{L^1(\Omega_T)} + ||g(u_n)S_l'(u_n)||_{L^1(\Omega_T)} + ||S_l'(u_n)\mu_n||_{L^1(\Omega_T)}.$$

Since $|S_l'(u_n)| \leq c_3\chi_{[-2l,2l]}(u_n)$ and $|S_l''(u_n)| \leq c_4|u_n|^{m-1}\chi_{[-2l,2l]}(u_n)$, we obtain

$$||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))}$$
$$\leq c_5 \left(||\nabla T_{(2l)^m}(|u_n|^{m-1}u_n)||_{L^2(\Omega_T)} + ||g||_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu_n|(\Omega_T) \right).$$

So from (5.2.14) we deduce that $\{(S_l(u_n))_t\}$ is bounded in $L^1(\Omega_T) + L^2((0,T);H^{-1}(\Omega))$ and for any $n \in \mathbb{N}$,

$$||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))} \leq c_5 \left((2l)^{m/2}(|\mu|(\Omega_T))^{1/2} + ||g||_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu|(\Omega_T) \right).$$

Moreover, $\{S_l(u_n)\}$ is bounded in $L^2(0,T,H_0^1(\Omega))$. Hence, $\{S_l(u_n)\}$ is relatively compact in $L^1(\Omega_T)$ for any $l>0$. Thanks to (5.2.15) we find

$$\begin{aligned}|\{||u_{n_1}|^m u_{n_1}-|u_{n_1}|^m u_{n_1}|>\ell\}| &\leq |\{|u_{n_1}|>l\}|+|\{|u_{n_2}|>l\}|+|\{|S_l(u_{n_1})-S_l(u_{n_2})|>\ell\}|\\ &\leq 2c_2 l^{-\frac{2}{N}-m}|\mu|(\Omega_T)^{\frac{N+2}{N}}+|\{|S_l(u_{n_1})-S_l(u_{n_2})|>\ell\}|.\end{aligned}$$

Thus, up to a subsequence $\{u_n\}$ converges a.e in Ω_T to a function u. Consequently, u is a very weak solution of equation (5.2.10) and satisfies (5.2.13) and (5.2.7)-(5.2.8). The other conclusions follow in the same way. ∎

Remark 5.2.7 *If $supp(\mu)\subset\overline{\Omega}\times[a,T]$ for $a>0$, then the solution u in Lemma 5.2.6 satisfies $u=0$ in $\Omega\times[0,a)$.*

Now we recall the important property of Radon measures which was proved in [6] and [35].

Proposition 5.2.8 *Let $s>1$ and $\mu\in\mathfrak{M}_b^+(\Omega_T)$. If μ is absolutely continuous with respect to $Cap_{2,1,s'}$ in Ω_T, there exists a nondecreasing sequence $\{\mu_n\}\subset\mathfrak{M}_b^+(\Omega_T)$, with compact support in Ω_T which converges to μ weakly in $\mathfrak{M}_b(\Omega_T)$ and satisfies $\mathbb{I}_2^R[\mu_n]\in L^s(\mathbb{R}^{N+1})$ for all $R>0$.*

Next we prove Theorem 5.1.3 in several steps of approximation :

Proof of Theorem 5.1.3. First suppose $m>1$. Assume that μ,σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. Then $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+,\sigma^-\otimes\delta_{\{t=0\}}+\mu^-$ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in $\Omega\times(-T,T)$. Applying Proposition 5.2.8 to $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+,\sigma^-\otimes\delta_{\{t=0\}}+\mu^-$, there exist two nondecreasing sequences $\{\upsilon_{1,n}\}$ and $\{\upsilon_{2,n}\}$ of positive bounded measures with compact support in $\Omega\times(-T,T)$ which converge respectively to $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+$ and $\sigma^-\otimes\delta_{\{t=0\}}+\mu^-$ in $\mathfrak{M}_b(\Omega\times(-T,T))$ and such that $\mathbb{I}_2^{2d}[\upsilon_{1,n}],\mathbb{I}_2^{2d}[\upsilon_{2,n}]\in L^q(\Omega\times(-T,T))$ for all $n\in\mathbb{N}$. By Lemma 5.2.6, there exists a sequence $\{u_{n_1,n_2,k_1,k_2}\}$ of of weak solution of the problems

$$\begin{cases}(u_{n_1,n_2,k_1,k_2})_t-\Delta(|u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2})+T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q)\\ \qquad\qquad -T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q)=\upsilon_{1,n_1}-\upsilon_{2,n_2}\ \text{ in }\Omega\times(-T,T),\\ u_{n_1,n_2,k_1,k_2}=0\quad\text{on }\partial\Omega\times(-T,T),\\ u_{n_1,n_2,k_1,k_2}(-T)=0\quad\text{in }\ \Omega,\end{cases}$$

which satisfy

$$|u_{n_1,n_2,k_1,k_2}|\leq C\left(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_1}+|\sigma|(\Omega)+|\mu|(\Omega_T)+1+\mathbb{I}_2^{2d}[\upsilon_{1,n_1}+\upsilon_{2,n_2}]\right), \tag{5.2.16}$$

and

$$\int_{\Omega_T}T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q)dxdt+\int_{\Omega_T}T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q)dxdt\leq|\mu|(\Omega_T).$$

Moreover, for any $n_1\in\mathbb{N},k_2>0$, $\{u_{n_1,n_2,k_1,k_2}\}_{n_2,k_1}$ is nonincreasing and for any $n_2\in\mathbb{N},k_1>0$, $\{u_{n_1,n_2,k_1,k_2}\}_{n_1,k_2}$ is nondecreasing. Therefore, thanks to the fact that $\mathbb{I}_2^{2d}[\upsilon_{1,n}]$,

$\mathbb{I}_2^{2d}[\upsilon_{2,n}] \in L^q(\Omega\times(-T,T))$ and from (5.2.16) and the dominated convergence Theorem, we deduce that $u_{n_1,n_2} = \lim\limits_{k_1\to\infty}\lim\limits_{k_2\to\infty} u_{n_1,n_2,k_1,k_2}$ is a very weak solution of

$$\begin{cases} (u_{n_1,n_2})_t - \Delta(|u_{n_1,n_2}|^{m-1}u_{n_1,n_2}) + |u_{n_1,n_2}|^{q-1}u_{n_1,n_2} = \upsilon_{1,n_1} - \upsilon_{2,n_2} & \text{in } \Omega\times(-T,T), \\ u_{n_1,n_2} = 0 \quad \text{on } \partial\Omega\times(-T,T), \\ u_{n_1,n_2}(-T) = 0 \quad \text{in } \Omega. \end{cases}$$

And (5.2.16) is true when u_{n_1,n_2,k_1,k_2} is replaced by u_{n_1,n_2}. Note that $\{u_{n_1,n_2}\}_{n_2}$ is non-increasing, $\{u_{n_1,n_2}\}_{n_1}$ is non-decreasing and

$$\int_{\Omega_T} |u_{n_1,n_2}|^q dxdt \le |\mu|(\Omega_T) \quad \forall\ n_1,n_2\in\mathbb{N}.$$

From the monotone convergence Theorem we obtain that $u = \lim\limits_{n_2\to\infty}\lim\limits_{n_1\to\infty} u_{n_1,n_2}$ is a very weak solution of

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \sigma\otimes\delta_{\{t=0\}} + \chi_{\Omega_T}\mu & \text{in } \Omega\times(-T,T), \\ u = 0 \quad \text{on } \partial\Omega\times(-T,T), \\ u(-T) = 0 \quad \text{in } \Omega. \end{cases}$$

which $u=0$ in $\Omega\times(-T,0)$ and u satisfies (5.1.3). Clearly, u is a very weak solution of equation (5.1.1).
Next suppose $m\le 1$. The proof is similar, with the new capacitary assumptions and (5.1.3) is replaced by (5.1.4). ■

We also obtain the subcritical case.

Theorem 5.2.9 *Let $m > \frac{N-2}{N}$ and $0 < q < m + \frac{2}{N}$. Then problem* (5.1.1) *has a very weak solution for any $\mu\in\mathfrak{M}_b(\Omega_T)$ and $\sigma\in\mathfrak{M}_b(\Omega)$.*

Proof. As the proof of Theorem 5.1.3, we can reduce to the case $\sigma = 0$. By Lemma 5.2.6, there exists a very weak solution u_{k_1,k_2} of

$$\begin{cases} (u_{k_1,k_2})_t - \Delta(|u_{k_1,k_2}|^{m-1}u_{k_1,k_2}) + T_{k_1}((u^+_{k_1,k_2})^q) - T_{k_2}((u^-_{k_1,k_2})^q) = \mu & \text{in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u_n(0) = 0 \quad \text{in } \Omega. \end{cases}$$

such that $\{u_{k_1,k_2}\}_{k_1}$ and $\{u_{k_1,k_2}\}_{k_2}$ are monotone sequences and

$$||u_{k_1,k_2}||_{L^{m+2/N,\infty}(\Omega_T)} \le C(|\mu|(\Omega_T))^{\frac{N+2}{mN+2}}.$$

In particular, $\{u_{k_1,k_2}\}$ is a uniformly bounded in $L^s(\Omega_T)$ for any $0<s<m+\frac{2}{N}$.
Therefore, we get that $u = \lim\limits_{k_2\to\infty}\lim\limits_{k_1\to\infty} u_{k_1,k_2}$ is a very weak solution of (5.1.1). ■

Next, from an idea of [8, Theorem 2.3], we obtain an existence result *for measures which present a good behaviour in time* :

Theorem 5.2.10 *Let $m > \frac{N-2}{N}$, $q > \max(1,m)$ and $f \in L^1(\Omega_T)$, $\mu \in \mathfrak{M}_b(\Omega_T)$, such that*

$$|\mu| \leq \omega \otimes F \quad \text{for some } \omega \in \mathfrak{M}_b^+(\Omega) \text{ and } F \in L^1_+((0,T)).$$

If ω is absolutely continuous with respect to the capacity $Cap_{\mathbf{G}_2, \frac{q}{q-m}}$ in Ω, then there exists a very weak solution to problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = f + \mu & \text{in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0. \end{cases} \tag{5.2.17}$$

Proof. For $R \in (0,\infty]$, we define the R-truncated Riesz elliptic potential of a measure $\nu \in \mathfrak{M}_b^+(\Omega)$ by

$$\mathbf{I}_2^R[\nu](x) = \int_0^R \frac{\nu(B_\rho(x))}{\rho^{N-2}} \frac{d\rho}{\rho} \quad \forall x \in \Omega.$$

By [9, Theorem 2.6],there exists a nondecreasing sequence $\{\omega_n\} \subset \mathfrak{M}_b^+(\Omega)$ with compact support in Ω which converges to ω in $\mathfrak{M}_b(\Omega)$ and such that $\mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\omega_n] \in L^{q/m}(\Omega)$ for any $n \in \mathbb{N}$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-,$$

and $\mu^+, \mu^- \leq \omega \otimes F$. We set

$$\mu_{1,n} = T_n(f^+) + \inf\{\mu^+, \omega_n \otimes T_n(F)\}, \qquad \mu_{2,n} = T_n(f^-) + \inf\{\mu^-, \omega_n \otimes T_n(F)\}.$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences converging to μ_1, μ_2 respectively in $\mathfrak{M}_b(\Omega_T)$ and $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $\mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\tilde{\omega}_n] \in L^{q/m}(\Omega)$. As in the proof of Theorem 5.1.3, there exists a sequence of weak solution $\{u_{n_1,n_2,k_1,k_2}\}$ of equations

$$\begin{cases} (u_{n_1,n_2,k_1,k_2})_t - \Delta(|u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2}) + T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q) \\ \qquad -T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q) = \mu_{1,n_1} - \mu_{2,n_2} \text{ in } \Omega_T, \\ u_{n_1,n_2,k_1,k_2} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{n_1,n_2,k_1,k_2}(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{5.2.18}$$

Using the comparison principle as in [8], we can assume that

$$-v_{n_2} \leq |u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2} \leq v_{n_1},$$

where for any $n \in \mathbb{N}$, v_n is a nonnegative weak solution of

$$\begin{cases} -\Delta v_n = \tilde{\omega}_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$v_n \leq c_1 \mathbf{I}_2^{2\mathrm{diam}(\Omega)}[\tilde{\omega}_n] \quad \forall\, n \in \mathbb{N}.$$

Hence, utilizing the arguments in the proof of Theorem 5.1.3, it is easy to obtain the result as desired. ∎

It is easy to show that $\omega \otimes \chi_{[0,T]}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}$ in Ω_T if any only if ω is absolutely continuous with respect to the capacity $\mathrm{Cap}_{\mathbf{G}_2,\frac{q}{q-m}}$ in Ω. Consequently, we obtain the following :

Corollary 5.2.11 *Let $m > \frac{N-2}{N}$, $q > \max(1,m)$ and $\omega \in \mathfrak{M}_b(\Omega)$. Then, ω is absolutely continuous with respect to the capacities $Cap_{\mathbf{G}_2,\frac{q}{q-m}}$ in Ω if and only if there exists a very weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \omega \otimes \chi_{[0,T]} \quad \text{in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0 \quad \text{in } \ \Omega. \end{cases} \tag{5.2.19}$$

5.3 $p-$Laplacian evolution equation

Here we consider solutions in the week sense of distributions, or in the renormalized sense,.

5.3.1 Distribution solutions

Definition 5.3.1 *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$ and $B \in C(\mathbb{R})$. A measurable function u is a distribution solution to problem* (5.3.1) *if $u \in L^s(0,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, p - \frac{N}{N+1}\right)$, and $B(u) \in L^1(\Omega_T)$, such that*

$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} |\nabla u|^{p-2}\nabla u.\nabla\varphi dxdt + \int_{\Omega_T} B(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0) d\sigma,$$

for every $\varphi \in C_c^1(\Omega \times [0,T))$.

Remark 5.3.2 *Let $\sigma' \in \mathfrak{M}_b(\Omega)$ and $a' \in (0,T)$, set $\omega = \mu + \sigma' \otimes \delta_{\{t=a'\}}$. Let u is a distribution solution to problem* (5.3.1) *with data ω and $\sigma = 0$, such that supp$(\mu) \subset \overline{\Omega} \times [a',T]$, and $u = 0, B(u) = 0$ in $\Omega \times (0,a')$. Then $\tilde{u} := u|_{\Omega\times[a',T)}$ is a distribution solution to problem* (5.3.1) *in $\Omega \times (a',T)$ with data μ and σ'.*

5.3.2 Renormalized solutions

The notion of renormalized solution is stronger. It was first introduced by Blanchard and Murat [11] to obtain uniqueness results for the p-Laplace evolution problem for L^1 data μ and σ, and developed by Petitta [36] for measure data μ. It requires a decomposition of the measure μ, that we recall now.

Let $\mathfrak{M}_0(\Omega_T)$ be the space of Radon measures in Ω_T which are absolutely continuous with respect to the C_p-capacity, defined at (5.1.6), and $\mathfrak{M}_s(\Omega_T)$ be the space of measures in Ω_T with support on a set of zero C_p-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega_T)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega_T) \cap \mathfrak{M}_b(\Omega_T)$ and $\mu_s \in \mathfrak{M}_s(\Omega_T)$. In turn μ_0 can be decomposed under the form

$$\mu_0 = f - \text{div } g + h_t,$$

where $f \in L^1(\Omega_T)$, $g \in (L^{p'}(\Omega_T))^N$ and $h \in L^p(0,T;W_0^{1,p}(\Omega))$, see [21]; and we say that (f,g,h) is a decomposition of μ_0. We say that a sequence of $\{\mu_n\}$ in $\mathfrak{M}_b(\Omega_T)$ converges to

$\mu \in \mathfrak{M}_b(\Omega_T)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty}\int_{\Omega_T}\varphi d\mu_n=\int_{\Omega_T}\varphi d\mu \quad \forall\varphi\in C(\Omega_T)\cap L^\infty(\Omega_T).$$

We recall that if u is a measurable function defined and finite a.e. in Ω_T, such that $T_k(u)\in L^p(0,T,W_0^{1,p}(\Omega))$ for any $k>0$, there exists a measurable function $w:\Omega_T\to\mathbb{R}^N$ such that $\nabla T_k(u)=\chi_{|u|\le k}w$ a.e. in Ω_T and for all $k>0$. We define the gradient ∇u of u by $w=\nabla u$.

Definition 5.3.3 *Let $p>\frac{2N+1}{N+1}$ and $\mu=\mu_0+\mu_s\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in L^1(\Omega)$ and $B\in C(\mathbb{R})$. A measurable function u is a renormalized solution of*

$$\begin{cases} u_t-\Delta_p u+B(u)=\mu \ \textit{in}\ \Omega_T,\\ u=0 \qquad \textit{on}\ \ \partial\Omega\times(0,T),\\ u(0)=\sigma \quad \textit{in}\ \ \Omega,\end{cases}\tag{5.3.1}$$

if there exists a decomposition (f,g,h) of μ_0 such that

$$\begin{aligned} &v=u-h\in L^s((0,T);W_0^{1,s}(\Omega))\cap L^\infty((0,T);L^1(\Omega)),\ \forall s\in\left[1,p-\frac{N}{N+1}\right),\\ &T_k(v)\in L^p((0,T);W_0^{1,p}(\Omega))\ \forall k>0, B(u)\in L^1(\Omega_T),\end{aligned}\tag{5.3.2}$$

and :

(i) for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0)=0$,

$$\begin{aligned} &-\int_\Omega S(\sigma)\varphi(0)dx-\int_{\Omega_T}\varphi_t S(v)dxdt+\int_{\Omega_T}S'(v)|\nabla u|^{p-2}\nabla u\nabla\varphi dxdt\\ &+\int_{\Omega_T}S''(v)\varphi|\nabla u|^{p-2}\nabla u\nabla v dxdt+\int_{\Omega_T}S'(v)\varphi B(u)dxdt=\int_{\Omega_T}(fS'(v)\varphi+g.\nabla(S'(v)\varphi)dxdt\end{aligned}\tag{5.3.3}$$

for any $\varphi\in L^p((0,T);W_0^{1,p}(\Omega))\cap L^\infty(\Omega_T)$ such that $\varphi_t\in L^{p'}((0,T);W^{-1,p'}(\Omega))+L^1(\Omega_T)$ and $\varphi(.,T)=0$;

(ii) for any $\phi\in C(\overline{\Omega_T})$,

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le v<2m\}}\phi|\nabla u|^{p-2}\nabla u\nabla v dxdt=\int_{\Omega_T}\phi d\mu_s^+ \quad \textit{and}\tag{5.3.4}$$

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{-m\ge v>-2m\}}\phi|\nabla u|^{p-2}\nabla u\nabla v dxdt=\int_{\Omega_T}\phi d\mu_s^-.\tag{5.3.5}$$

We first mention a convergence result of [7].

Proposition 5.3.4 *Let $\{\mu_n\}$ be bounded in $\mathfrak{M}_b(\Omega_T)$ and $\{\sigma_n\}$ be bounded in $L^1(\Omega)$, and $B \equiv 0$. Let u_n be a renormalized solution of (5.3.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$ and initial data σ_n. If $\{f_n\}$ is bounded in $L^1(\Omega_T)$, $\{g_n\}$ bounded in $(L^{p'}(\Omega_T))^N$ and $\{h_n\}$ convergent in $L^p(0,T,W_0^{1,p}(\Omega))$, then, up to a subsequence, $\{u_n\}$ converges to a function u in $L^1(\Omega_T)$. Moreover, if $\{\mu_n\}$ is bounded in $L^1(\Omega_T)$ then $\{u_n\}$ is convergent in $L^s(0,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, p - \frac{N}{N+1}\right)$.*

Next we recall the fundamental stability result of [7].

Theorem 5.3.5 *Suppose that $p > \frac{2N+1}{N+1}$ and $B \equiv 0$. Let $\sigma \in L^1(\Omega)$ and*

$$\mu = f - div\ g + h_t + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(\Omega_T),$$

with $f \in L^1(\Omega_T), g \in (L^{p'}(\Omega_T))^N$, $h \in L^p((0,T); W_0^{1,p}(\Omega))$ and $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega_T)$. Let $\sigma_n \in L^1(\Omega)$ and

$$\mu_n = f_n - div\ g_n + (h_n)_t + \rho_n - \eta_n \in \mathfrak{M}_b(\Omega_T),$$

with $f_n \in L^1(\Omega_T), g_n \in (L^{p'}(\Omega_T))^N, h_n \in L^p((0,T); W_0^{1,p}(\Omega))$, and $\rho_n, \eta_n \in \mathfrak{M}_b^+(\Omega_T)$, such that

$$\rho_n = \rho_n^1 - div\ \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - div\ \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega_T), \rho_n^2, \eta_n^2 \in (L^{p'}(\Omega_T))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathfrak{M}_s^+(\Omega_T)$.

Assume that $\{\mu_n\}$ is bounded in $\mathfrak{M}_b(\Omega_T)$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ converge to σ, f, g, h in $L^1(\Omega)$, weakly in $L^1(\Omega_T)$, in $(L^{p'}(\Omega_T))^N$,in $L^p(0,T,W_0^{1,p}(\Omega))$ respectively and $\{\rho_n\}, \{\eta_n\}$ converge to μ_s^+, μ_s^- in the narrow topology of measures; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(\Omega_T)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(\Omega_T))^N$.
Let $\{u_n\}$ be a sequence of renormalized solutions of

$$\begin{cases} (u_n)_t - \Delta_p u_n = \mu_n & in\ \Omega_T, \\ u_n = 0 & on\ \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n & in\ \Omega, \end{cases} \tag{5.3.6}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$.

Then up to a subsequence, $\{u_n\}$ converges a.e. in Ω_T to a renormalized solution u of (5.3.1), and $\{v_n\}$ converges a.e. in Ω_T to $v = u - h$. Moreover, $\{\nabla v_n\}$ converge to ∇v a.e in Ω_T, and $\{T_k(v_n)\}$ converges to $T_k(v)$ strongly in $L^p(0,T,W_0^{1,p}(\Omega))$ for any $k > 0$.

In order to apply this Theorem, we need some the following properties concerning approximate measures of $\mu \in \mathfrak{M}_b^+(\Omega_T)$, see also [7].

Proposition 5.3.6 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b^+(\Omega_T)$, $\mu_0 \in \mathfrak{M}_0(\Omega_T) \cap \mathfrak{M}_b^+(\Omega_T)$ and $\mu_s \in \mathfrak{M}_s(\Omega_T)$. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}^N, \mathbb{R}$ respectively. There exists a sequence of measures $\mu_{n,0} = (f_n, g_n, h_n)$, such that $f_n, g_n, h_n \in C_c^\infty(\Omega_T)$ and strongly converge to f, g, h in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, $\mu_{n,s} \in C_c^\infty(\Omega_T)$ converges to $\mu_s \in \mathfrak{M}_s^+(\Omega_T)$, and $\mu_n = \mu_{n,0} + \mu_{n,s}$ converges to μ, in the narrow topology, and satisfying $0 \leq \mu_n \leq (\varphi_{1,n}\varphi_{2,n}) * \mu$, and*

$$||f_n||_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + ||h_n||_{L^p(0,T,W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \leq 2\mu(\Omega_T) \quad for\ any\ \ n \in \mathbb{N}.$$

Proposition 5.3.7 *Let $\mu = \mu_0+\mu_s,\ \mu_n = \mu_{n,0}+\mu_{n,s} \in \mathfrak{M}_b^+(\Omega_T)$ with $\mu_0, \mu_{n,0} \in \mathfrak{M}_0(\Omega_T)\cap \mathfrak{M}_b^+(\Omega_T)$ and $\mu_{n,s}, \mu_s \in \mathfrak{M}_s^+(\Omega_T)$ such that $\{\mu_n\}$ is nondecreasing and converges to μ in $\mathfrak{M}_b(\Omega_T)$. Then, $\{\mu_{n,s}\}$ is nondecreasing and converging to μ_s in $\mathfrak{M}_b(\Omega_T)$; and there exist decompositions (f, g, h) of μ_0, (f_n, g_n, h_n) of $\mu_{n,0}$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, satisfying*

$$||f_n||_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + ||h_n||_{L^p((0,T);W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \le 2\mu(\Omega_T) \quad \textit{for any} \ \ n \in \mathbb{N}.$$

5.3.3 Proof of Theorem 5.1.5

Here the crucial point is a result of Liskevich, Skrypnik and Sobol [31] for the p-Laplace evolution problem without absorption :

Theorem 5.3.8 *Let $p > 2$, and $\mu \in \mathfrak{M}_b(\Omega_T)$. If $u \in C([0,T]; L^2_{loc}(\Omega))\cap L^p_{loc}(0,T, W^{1,p}_{loc}(\Omega))$ is a distribution solution to equation*

$$u_t - \Delta_p u = \mu \quad in\ \Omega_T,$$

then there exists $C = C(N, p)$ such that, for every Lebesgue point $(x,t) \in \Omega_T$ of u and any $\rho > 0$ such that $Q_{\rho,\rho^p}(x,t) := B_\rho(x) \times (t-\rho^p, t+\rho^p) \subset \Omega_T$ one has

$$|u(x,t)| \le C\left(1 + \left(\frac{1}{\rho^{N+p}}\int_{Q_{\rho,\rho^p}(x,t)} |u|^{(\lambda+1)(p-1)} dyds\right)^{\frac{1}{1+\lambda(p-1)}} + \mathbf{P}_p^\rho[\mu](x,t)\right), \quad (5.3.7)$$

where $\lambda = \min\{1/(p-1), 1/N\}$ and

$$\mathbf{P}_p^\rho[\mu](x,t) = \sum_{i=0}^{\infty} D_p(\rho_i)(x,t),$$

$$D_p(\rho_i)(x,t) = \inf_{\tau>0}\left\{(p-2)\tau^{-\frac{1}{p-2}} + \frac{1}{2(p-1)^{p-1}}\frac{|\mu|(Q_{\rho_i,\tau\rho_i^p}(x,t))}{\rho_i^N}\right\},$$

with $\rho_i = 2^{-i}\rho$, $Q_{\rho,\tau\rho^p}(x,t) = B_\rho(x) \times (t-\tau\rho^p, t+\tau\rho^p)$.

As a consequence, we deduce the following estimate :

Proposition 5.3.9 *If u is a distribution solution of problem*

$$\begin{cases} u_t - \Delta_p u = \mu \ in\ \Omega_T, \\ u = 0 \qquad on\ \partial\Omega \times (0,T), \\ u(0) = 0 \quad in\ \Omega, \end{cases}$$

with data $\mu \in C_b(\Omega_T)$. Then there exists $C = C(N, p)$ such that for a.e. $(x,t) \in \Omega_T$,

$$|u(x,t)| \le C\left(1 + D + \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu|](x,t)\right), \quad (5.3.8)$$

where m_3 and D are defined at (5.1.8).

Proof. Let $x_0 \in \Omega$ and $Q = B_{2D}(x_0) \times (-(2D)^p, (2D)^p)$.
Let $U \in C(Q) \cap L^p((-(2D)^p, (2D)^p); W_0^{1,p}(B_{2D}(x_0)))$ be the distribution solution of

$$\begin{cases} U_t - \Delta_p U = \chi_{\Omega_T}|\mu| & \text{in } Q, \\ u = 0 & \text{on } \partial B_{2D}(x_0) \times (-(2D)^p, (2D)^p), \\ u(-(2D)^p) = 0 & \text{in } \ B_{2D}(x_0), \end{cases} \tag{5.3.9}$$

where for $x_0 \in \Omega$. Thus, by Theorem 5.3.8 we have, for any $(x,t) \in \Omega_T$,

$$U(x,t) \leq c_1 \left(1 + \left(\frac{1}{D^{N+p}} \int_{Q_{D,D^p}(x,t)} |U|^{(\lambda+1)(p-1)} dyds \right)^{\frac{1}{1+\lambda(p-1)}} + \mathbf{P}_p^D[\mu](x,t) \right), \tag{5.3.10}$$

where $Q_{D,D^p}(x,t) = B_D(x) \times (t - D^p, t + D^p)$.

According to Proposition 2.8 and Remark 2.9 of [7], there exists a constant $C_2 > 0$ such that

$$|\{|U| > \ell\}| \leq c_2(|\mu|(\Omega_T))^{\frac{p+N}{N}} \ell^{-p+1-\frac{p}{N}} \qquad \forall \ell > 0.$$

Thus, for any $\ell_0 > 0$,

$$\begin{aligned} \int_Q |U|^{(\lambda+1)(p-1)} dxdt &= (\lambda+1)(p-1) \int_0^\infty \ell^{(\lambda+1)(p-1)-1} |\{|U| > \ell\}| d\ell \\ &= (\lambda+1)(p-1) \left(\int_0^{\ell_0} \ell^{(\lambda+1)(p-1)-1} |\{|U| > \ell\}| d\ell + \int_{\ell_0}^\infty \ell^{(\lambda+1)(p-1)-1} |\{|U| > \ell\}| d\ell \right) \\ &\leq c_3 D^{N+p} \ell_0^{(\lambda+1)(p-1)} + c_4 \ell_0^{(\lambda+1)(p-1)-p+1-\frac{p}{N}} (|\mu|(\Omega_T))^{\frac{p+N}{N}}. \end{aligned}$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{\frac{N+p}{(p-1)N+p}}$, we get

$$\int_Q |U|^{(\lambda+1)(p-1)} dxdt \leq c_5 D^{N+p} \left(\frac{|\mu|(\Omega_T)}{D^N} \right)^{\frac{(N+p)(\lambda+1)(p-1)}{(p-1)N+p}}. \tag{5.3.11}$$

Next we show that

$$\mathbf{P}_p^D[\mu](x,t) \leq (p-2)D + c_6 \mathbb{I}_2^{2D}[|\mu|](x,t). \tag{5.3.12}$$

Indeed, we have

$$D_p(\rho_i)(x,t) \leq (p-2)\rho_i + \frac{1}{2(p-1)^{p-1}} \frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N},$$

where $\rho_i = 2^{-i} D$. Thus,

$$\begin{aligned} \mathbf{P}_p^D[\mu](x,t) &\leq (p-2)D + \frac{1}{2(p-1)^{p-1}} \sum_{i=0}^\infty \frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N} \\ &\leq (p-2)D + c_7 \int_0^{2D} \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho}. \end{aligned}$$

So from (5.3.11), (5.3.12) and (5.3.10) we get, for any $(x,t)\in\Omega_T$,

$$|U(x,t)|\leq c_8\left(1+D+\left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3}+\mathbb{I}_2^{2D}[|\mu|](x,t)\right).$$

By the comparison principle we get $|u|\leq U$ in Ω_T, thus (5.3.8) follows. ■

Proposition 5.3.10 *Let $p>2$, and $\mu\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in\mathfrak{M}_b(\Omega)$. There exists a distribution solution u of problem*

$$\begin{cases} u_t-\Delta_p u=\mu & in\ \Omega_T,\\ u=0 & on\ \partial\Omega\times(0,T),\\ u(0)=\sigma. \end{cases}\tag{5.3.13}$$

which satisfies for any $(x,t)\in\Omega_T$

$$|u(x,t)|\leq C\left(1+D+\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{D^N}\right)^{m_3}+\mathbb{I}_2^{2D}\left[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|\right](x,t)\right),\tag{5.3.14}$$

where $C=C(N,p)$. Moreover, if $\sigma\in L^1(\Omega)$, u is a renormalized solution.

Proof. Let $\{\varphi_{1,n}\},\{\varphi_{2,n}\}$ be sequences of standard mollifiers in $\mathbb{R}^N$ and $\mathbb{R}$. Let $\mu=\mu_0+\mu_s\in\mathfrak{M}_b(\Omega_T)$, with $\mu_0\in\mathfrak{M}_0(\Omega_T)$, $\mu_s\in\mathfrak{M}_s(\Omega_T)$. By Lemma 5.3.6, there exist sequences of nonnegative measures $\mu_{n,0,i}=(f_{n,i},g_{n,i},h_{n,i})$ and $\mu_{n,s,i}$ such that $f_{n,i},g_{n,i},h_{n,i}\in C_c^\infty(\Omega_T)$ and strongly converge to some f_i,g_i,h_i in $L^1(\Omega_T),(L^{p'}(\Omega_T))^N$ and $L^p((0,T);W_0^{1,p}(\Omega))$ respectively, and $\mu_{n,1},\mu_{n,2},\mu_{n,s,1},\mu_{n,s,2}\in C_c^\infty(\Omega_T)$ converge to $\mu^+,\mu^-,\mu_s^+,\mu_s^-$ in the narrow topology, with $\mu_{n,i}=\mu_{n,0,i}+\mu_{n,s,i}$, for $i=1,2$, and satisfying

$$\mu_0^+=(f_1,g_1,h_1),\mu_0^-=(f_2,g_2,h_2)\ \text{ and }\ 0\leq\mu_{n,1}\leq(\varphi_{1,n}\varphi_{2,n})*\mu^+,0\leq\mu_{n,2}\leq(\varphi_{1,n}\varphi_{2,n})*\mu^-.$$

Let $\sigma_{1,n},\sigma_{2,n}\in C_c^\infty(\Omega)$ converge to σ^+ and σ^- in the narrow topology, and in $L^1(\Omega)$ if $\sigma\in L^1(\Omega)$, such that

$$0\leq\sigma_{1,n}\leq\varphi_{1,n}*\sigma^+,0\leq\sigma_{2,n}\leq\varphi_{1,n}*\sigma^-.$$

Set $\mu_n=\mu_{n,1}-\mu_{n,2}$ and $\sigma_n=\sigma_{1,n}-\sigma_{2,n}$.
Let u_n be solution of the approximate problem

$$\begin{cases} (u_n)_t-\Delta_p u_n=\mu_n & \text{in } \Omega_T,\\ u_n=0 & \text{on } \partial\Omega\times(0,T),\\ u_n(0)=\sigma_n & \text{on } \Omega. \end{cases}\tag{5.3.15}$$

Let $g_{n,m}(x,t)=\sigma_n(x)\int_{-T}^t\varphi_{2,m}(s)ds$. As in proof of Theorem 2.1 in [35], by Theorem 5.3.5, there exists a sequence $\{u_{n,m}\}_m$ of solutions of the problem

$$\begin{cases} (u_{n,m})_t-\Delta_p u_{n,m}=(g_{n,m})_t+\chi_{\Omega_T}\mu_n & \text{in } \Omega\times(-T,T),\\ u_{n,m}=0 & \text{on } \partial\Omega\times(-T,T),\\ u_{n,m}(-T)=0 & \text{on } \Omega, \end{cases}\tag{5.3.16}$$

which converges to u_n in $\Omega \times (0,T)$. By Proposition 5.3.9, there holds, for any $(x,t) \in \Omega_T$,

$$|u_{n,m}(x,t)| \le c_1 \left(1 + D + \left(\frac{|\mu_n|(\Omega_T) + (|\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T,T))}{D^N}\right)^{m_3}\right) + c_1 \mathbb{I}_2^{2D}[|\mu_n| + |\sigma_n| \otimes \varphi_{2,m}](x,t).$$

Therefore

$$|u_{n,m}(x,t)| \le c_1 \left(1 + D + \left(\frac{|\mu_n|(\Omega_T) + (|\sigma_n| \otimes \varphi_{2,m})(\Omega \times (-T,T))}{D^N}\right)^{m_3}\right) + c_1 (\varphi_{1,n}\varphi_{2,m}) * \mathbb{I}_2^{2D}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](x,t).$$

Letting $m \to \infty$, we get

$$|u_n(x,t)| \le c_1 \left(1 + D + \left(\frac{|\mu_n|(\Omega_T) + |\sigma_n|(\Omega)}{D^N}\right)^{m_3}\right) + c_1(\varphi_{1,n}) * (\mathbb{I}_2^{2D}[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}](.,t))(x).$$

Therefore, by Proposition 5.3.4 and Theorem 5.3.5 , up to a subsequence, $\{u_n\}$ converges to a distribution solution u of (5.3.13) (a renormalized solution if $\sigma \in L^1(\Omega)$), and satisfying (5.3.14). ■

Proof of Theorem 5.1.5. Step 1. First, assume that $\sigma \in L^1(\Omega)$. Because μ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,q'}$, so are μ^+ and μ^-. Applying Proposition 5.2.8 to μ^+, μ^-, there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathfrak{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2^{2D}[\mu_{1,n}], \mathbb{I}_2^{2D}[\mu_{2,n}] \in L^q(\Omega_T)$ for all $n \in \mathbb{N}$.

For $i = 1,2$, set $\tilde{\mu}_{i,1} = \mu_{i,1}$ and $\tilde{\mu}_{i,j} = \mu_{i,j} - \mu_{i,j-1} \ge 0$, so $\mu_{i,n} = \sum_{j=1}^n \tilde{\mu}_{i,j}$. We write

$$\mu_{i,n} = \mu_{i,n,0} + \mu_{i,n,s}, \tilde{\mu}_{i,j} = \tilde{\mu}_{i,j,0} + \tilde{\mu}_{i,j,s}, \quad \text{with } \mu_{i,n,0}, \tilde{\mu}_{i,n,0} \in \mathfrak{M}_0(\Omega_T), \mu_{i,n,s}, \tilde{\mu}_{i,n,s} \in \mathfrak{M}_s(\Omega_T).$$

Let $\{\varphi_m\}$ be a sequence of mollifiers in $\mathbb{R}^{N+1}$. As in the proof of Proposition 5.3.10, for any $j \in \mathbb{N}$ and $i = 1,2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m,i,j,0} = (f_{m,i,j}, g_{m,i,j}, h_{m,i,j})$ and $\tilde{\mu}_{m,i,j,s}$ such that $f_{m,i,j}, g_{m,i,j}, h_{m,i,j} \in C_c^\infty(\Omega_T)$ strongly converge to some $f_{i,j}, g_{i,j}, h_{i,j}$ in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p(0,T,W_0^{1,p}(\Omega))$ respectively ; and $\tilde{\mu}_{m,i,j}$, $\tilde{\mu}_{m,i,j,s} \in C_c^\infty(\Omega_T)$ converge to $\tilde{\mu}_{i,j}, \tilde{\mu}_{i,j,s}$ in the narrow topology with $\tilde{\mu}_{m,i,j} = \tilde{\mu}_{m,i,j,0} + \tilde{\mu}_{m,i,j,s}$, which satisfy $\tilde{\mu}_{i,j,0} = (f_{i,j}, g_{i,j}, h_{i,j})$, and

$$0 \le \tilde{\mu}_{m,i,j} \le \varphi_m * \tilde{\mu}_{i,j}, \tilde{\mu}_{m,i,j}(\Omega_T) \le \tilde{\mu}_{i,j}(\Omega_T),$$
$$||f_{m,i,j}||_{L^1(\Omega_T)} + ||g_{m,i,j}||_{(L^{p'}(\Omega_T))^N} + ||h_{m,i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))} + \mu_{m,i,j,s}(\Omega_T) \le 2\tilde{\mu}_{i,j}(\Omega_T). \tag{5.3.17}$$

Note that, for any $n, m \in \mathbb{N}$,

$$\sum_{j=1}^n (\tilde{\mu}_{m,1,j} + \tilde{\mu}_{m,2,j}) \le \varphi_m * (\mu_{1,n} + \mu_{2,n}) \text{ and } \sum_{j=1}^n (\tilde{\mu}_{m,1,j}(\Omega_T) + \tilde{\mu}_{m,2,j}(\Omega_T)) \le |\mu|(\Omega_T).$$

For any $n, k, m \in \mathbb{N}$, let $u_{n,k,m}, v_{n,k,m} \in W$ be solutions of problems

$$\begin{cases} (u_{n,k,m})_t - \Delta_p u_{n,k,m} + T_k(|u_{n,k,m}|^{q-1}u_{n,k,m}) = \sum_{j=1}^n (\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}) & \text{in } \Omega_T, \\ u_{n,k,m} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_{n,k,m}(0) = T_n(\sigma^+) - T_n(\sigma^-) & \text{on } \Omega, \end{cases} \tag{5.3.18}$$

and

$$\begin{cases} (v_{n,k,m})_t - \Delta_p v_{n,k,m} + T_k(v_{n,k,m}^q) = \sum_{j=1}^n (\tilde{\mu}_{m,1,j} + \tilde{\mu}_{m,2,j}) \text{ in } \Omega_T, \\ v_{n,k,m} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ v_{n,k,m}(0) = T_n(|\sigma|) \quad \text{on } \Omega. \end{cases} \tag{5.3.19}$$

By the comparison principle and Proposition 5.3.10 we have for any m, k the sequences $\{v_{n,k,m}\}_n$ is increasing and

$$|u_{n,k,m}| \le v_{n,k,m} \le c_1 \left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|) \otimes \delta_{\{t=0\}}\right]\right)$$
$$+ c_1 \varphi_m * \mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right].$$

Moreover,

$$\int_{\Omega_T} T_k(v_{n,k,m}^q) dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

As in [8, Proof of Lemma 5.3], thanks to Proposition 5.3.4 and Theorem 5.3.5, up to subsequences, $\{u_{n,k,m}\}_m$ converges to a renormalized solutions $u_{n,k}$ of problem

$$\begin{cases} (u_{n,k})_t - \Delta_p u_{n,k} + T_k(|u_{n,k}|^{q-1} u_{n,k}) = \mu_{1,n} - \mu_{2,n} \text{ in } \Omega_T, \\ u_{n,k} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_{n,k}(0) = T_n(\sigma^+) - T_n(\sigma^-) \quad \text{on } \Omega, \end{cases}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j} - \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} - \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} - \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0} - \mu_{2,n,0}$; and $\{v_{n,k,m}\}_m$ converges to a solution $v_{n,k}$ of

$$\begin{cases} (v_{n,k})_t - \Delta_p v_{n,k} + T_k(v_{n,k}^q) = \mu_{1,n} + \mu_{2,n} \text{ in } \Omega_T, \\ v_{n,k} = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ v_{n,k}(0) = T_n(|\sigma|) \quad \text{on } \Omega. \end{cases}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j} + \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} + \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} + \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0} + \mu_{2,n,0}$. And there holds

$$|u_{n,k}| \le v_{n,k} \le c_1 \left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|) \otimes \delta_{\{t=0\}}\right]\right)$$
$$+ c_1 \mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right].$$

Observe that $\mathbb{I}_2^{2D}[\mu_{1,n} + \mu_{2,n}] \in L^q(\Omega_T)$ for any $n \in \mathbb{N}$. Then, as in [8, Proof of Lemma 5.4], thanks to Proposition 5.3.4 and Theorem 5.3.5, up to a subsequence, $\{u_{n,k}\}_k$ $\{v_{n,k}\}_k$ converge to renormalized solutions u_n, v_n of problems

$$\begin{cases} (u_n)_t - \Delta_p u_n + |u_n|^{q-1} u_n = \mu_{1,n} - \mu_{2,n} \text{ in } \Omega_T, \\ u_n = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = T_n(\sigma^+) - T_n(\sigma^-) \quad \text{in } \Omega, \end{cases} \tag{5.3.20}$$

$$\begin{cases} (v_n)_t - \Delta_p v_n + v_n^q = \mu_{1,n} + \mu_{2,n} \text{ in } \Omega_T, \\ v_n = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ v_n(0) = T_n(|\sigma|) \quad \text{in } \Omega, \end{cases} \tag{5.3.21}$$

which still satisfy

$$|u_n| \le v_n \le c_1\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|)\otimes\delta_{\{t=0\}}\right]\right) + c_1\mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right].$$

and the sequence $\{v_n\}_n$ is increasing and

$$\int_{\Omega_T} v_n^q dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

Note that from (5.3.17) we have

$$||f_{i,j}||_{L^1(\Omega_T)} + \|g_{i,j}\|_{(L^{p'}(\Omega_T))^N} + ||h_{i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))} \le 2\tilde{\mu}_{i,j}(\Omega_T),$$

which implies

$$||\sum_{j=1}^n f_{i,j}||_{L^1(\Omega_T)} + ||\sum_{j=1}^n g_{i,j}||_{(L^{p'}(\Omega_T))^N} + ||\sum_{j=1}^n h_{i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))} \le 2\mu_{i,n}(\Omega_T) \le 2|\mu|(\Omega_T).$$

Finally, as in [8, Proof of Theorem 5.2], from Proposition 5.3.4, Theorem 5.3.5 and the monotone convergence Theorem, up to subsequences $\{u_n\}_n$, $\{v_n\}_n$ converge to a renormalized solutions u, v of problem

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \quad \text{in} \quad \Omega_T, \\ u = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega\times(0,T), \\ u(0) = \sigma \quad \text{in} \quad \Omega, \end{cases}$$

relative to the decomposition $(\sum_{j=1}^\infty f_{1,j} - \sum_{j=1}^\infty f_{2,j}, \sum_{j=1}^\infty g_{1,j} - \sum_{j=1}^\infty g_{2,j}, \sum_{j=1}^\infty h_{1,j} - \sum_{j=1}^\infty h_{2,j})$ of μ_0, and

$$\begin{cases} v_t - \Delta_p v + v^q = |\mu| \quad \text{in} \quad \Omega_T, \\ v = 0 \qquad\qquad\qquad \text{on } \partial\Omega\times(0,T), \\ v(0) = |\sigma| \quad \text{in} \quad \Omega, \end{cases}$$

relative to the decomposition $(\sum_{j=1}^\infty f_{1,j} + \sum_{j=1}^\infty f_{2,j}, \sum_{j=1}^\infty g_{1,j} + \sum_{j=1}^\infty g_{2,j}, \sum_{j=1}^\infty h_{1,j} + \sum_{j=1}^\infty h_{2,j})$ of $|\mu_0|$ respectively ; and

$$|u| \le v \le c_1\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma|\otimes\delta_{\{t=0\}} + |\mu|\right]\right)$$

Remark that, if $\sigma \equiv 0$ and $\operatorname{supp}(\mu) \subset \overline{\Omega}\times[a,T]$, $a > 0$, then $u = v = 0$ in $\Omega\times(0,a)$, since $u_{n,k} = v_{n,k} = 0$ in $\Omega\times(0,a)$.

Step 2. We consider any $\sigma \in \mathfrak{M}_b(\Omega)$ such that σ is absolutely continuous with respect to the capacity $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. So, $\mu + \sigma\otimes\delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\Omega\times(-T,T)$. As above, we verify that there exists a renormalized solution u of

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \chi_{\Omega_T}\mu + \sigma\otimes\delta_{\{t=0\}} \quad \text{in} \quad \Omega\times(-T,T) \\ u = 0 \qquad\qquad \text{on } \partial\Omega\times(-T,T), \\ u(-T) = 0 \quad \text{on} \quad \Omega, \end{cases}$$

satisfying $u = 0$ in $\Omega\times(-T,0)$ and (5.1.7). Finally, from Remark 5.3.2 we get the result. This completes the proof of the Theorem. ■

Bibliographie

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J., **42**, no. 4, 765-778 (1975).

[2] D. R. Adams, L.I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften 31, Springer-Verlag (1999).

[3] D.R. Adams, R.J. Bagby, *Translation-dilation invariant estimates for Riesz potentials*, Indiana Univ. Math. J., **23**, 1051-1067 (1974).

[4] R.J. Bagby, *Lebesgue spaces of parabolic potentials*, Ill. J. Math., **15**, 610-634 (1971).

[5] P. Baras and M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[6] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin., **3**, 185-212 (1985).

[7] M. F. Bidaut-Véron, Quoc-Hung Nguyen, *Stability properties for quasilinear parabolic equations with measure data*, to appear in Journal of European Mathematical Society, arXiv :1409.1518.

[8] M. F. Bidaut-Véron, Quoc-Hung Nguyen, *Evolution equations of p-Laplace type with absorption or source terms and measure data*, to appear in Communications in contemporary mathematics, arXiv :1409.1520.

[9] M. F. Bidaut-Véron, Q.H Nguyen, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math.Pures et Appl.,**102**, 315-337 (2014).

[10] L. Boccardo , A. Dall'Aglio, T. Gallouet and L. Orsina, *Nonlinear parabolic equations with measure data, J. Funct. Anal.* **147**, 237-258 (1997).

[11] D. Blanchard, F. Murat, *Renormalized solutions of nonlinear parabolic equation with L^1 data : existence and uniqueness*, Proc. Roy. Soc. Edinburgh, **127A**, 1153-1179 (1997).

[12] V. Bogelein, F. Duzaar and U. Gianazza, Inst. Mittag-Leffler, Report 31 (2013-2014).

[13] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions,* J.Math.Pures Appl., **62**, 73-97 (1983).

[14] E. Chasseigne,*Contribution à la Théorie des traces pour les équations quasilinéaires paraboliques,*, Univ. Tours, France (2000).

[15] E. Chasseigne, *Initial trace for a porous medium equation : I. The strong absorption case,* Ann. Mat. Pura Appl. IV, **179**, 413-458 (2001).

[16] E. Chasseigne and J.L. Vazquez, *Sets of admissible initialdata for porous-medium equation with absorption,* E.J.D.E., conf.**08**,, 53-83 (2001).

[17] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa, **28**, 741-808 (1999).

[18] A. Dall'Aglio and L. Orsina, *Existence results for some nonlinear parabolic equations with nonregular data*, Diff. Int. Equ., **5**, 1335-1354 (1992).

[19] P.Daskalopoulou and C. Kenig, *Degenerate diffusions. Initial value problems and local regularity theory*, EMS Tracts in Mathematics, **1**, European Mathematical Society Zurich (2007).

[20] E. DiBenedetto : *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York (1993).

[21] J. Droniou , A. Porretta and A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Anal. **19**, 99-161 (2003).

[22] F. Duzaar, G. Mingione, *Gradient estimate via non-linear potentials*, the American Journal of Mathematics, **133**, no. 4, 1093-1149 (2011).

[23] D. Feyel, A. de la Pradelle : *Topologies fines et compactifications associées à certains espaces de Dirichlet*, Ann. Inst. Fourier Grenoble,**27**, 121-146 (1977).

[24] J. Heinonen, T. Kilpelainen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, N.Y. (2006).

[25] T. Kilpelainen, P. Koskela : *Global integrability of the gradients of solutions to partial differential equations.* Nonlinear Anal., **23**, no. 7, 899-909 (1994).

[26] T. Kuusi, G. Mingione : *Riesz potentials and nonlinear parabolic equations,* to appear in Arch. for Rat. Mech. Anal. (2013)

[27] T. Kuusi, G. Mingione : *the Wolff gradient bound for degenerate parabolic equations* to appear in J. European Math. Soc.

[28] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. UralA-Ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, Amer. Math. Soc., Providence (1968).

[29] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial boundary-value problems,* Proc. Royal Soc. Edinburg Sect A, **89**, 217-237 (1981).

[30] G.M. Lieberman : *Second Order Parabolic Differential Equations*, World Scientific press, River Edge (1996).

[31] Liskevich, Vitali ; Skrypnik, Igor I. ; Sobol, Zeev *Estimates of solutions for the parabolic p-Laplacian equation with measure via parabolic nonlinear potentials.* Commun. Pure Appl. Anal., **12**, 1731-1744 (2013).

[32] Liskevich, Vitali ; Skrypnik, Igor I. *Pointwise estimates for solutions to the porous medium equation with measure as a forcing term.* Isreal J. of mathematics, **194**, 259-275 (2013).

[33] G. Mingione : *Nonlinear measure data problems* Milan J. Math., **79** 429-496 (2011).

[34] Q.H.Nguyen , L. Véron *Quasilinear and Hessian type equations with exponential reaction and measure data*, Archive for Rational Mechanics and Analysis, **214**, 235-267 (2014).

[35] Q.H.Nguyen, *Potential estimates and quasilinear equations with measure data*, arXiv :1405.2587v1.

[36] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data,* Ann. Math. Pura Appl., **187**, 563-604 (2008).

[37] F. Petitta, A. Ponce and A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ., **11**, 861-905 (2011).

[38] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailiov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin and New York (1995).

[39] J.L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations if porous medium type*, Oxford lecture series in mathematics ans its applications, 33, Oxford University Press, Oxford (2006).

[40] J.L.Vazquez, *The porous medium equation. Mathematical theory* Oxford Mathematical monographs, the clarendon, Oxford University Press, Oxford (2007).

Chapitre 6

Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

Abstract

We obtain sufficient conditions, expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities, for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bounded domain Ω when $q > p - 1 > 0$. We apply our results to equations (3) $-\Delta_p u + a|\nabla u|^q + bu^s = 0$, (4) $\Delta_p u + u^{-\gamma} = 0$ with $1 < p \le 2$, $1 \le q \le p$, $a > 0, b > 0$ and $q > p - 1$, $s \ge p - 1$, $\gamma > 0$.

6.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $1 < p \leq N$. We denote $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. In this paper we study some questions relative to the existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) = 0 \ \text{ in } \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty, \end{aligned} \tag{6.1.1}$$

where g is a continuous nondecreasing function vanishing at 0, and most often $g(u)$ is either $\operatorname{sign}(u)(e^{|u|} - 1)$ or $|u|^{q-1} u$ with $q > p - 1$. A solution to problem (6.1.1) is called a *large solution.* When the domain is regular in the sense that the Dirichlet problem with continuous boundary dataϕ

$$\begin{aligned} -\Delta_p u + g(u) = 0 \ \text{ in } \Omega, \\ u = \phi \ \text{ on } \partial\Omega, \end{aligned} \tag{6.1.2}$$

admits a solution, it is clear that problem (6.1.1) admits a solution. It is known that a necessary and sufficient condition for the solvability of problem (6.1.2) is the the *extended Wiener criterion,* due to Wiener [21] when $p = 2$ and Maz'ya [13], Kilpelainen and Maly [7] when $p \neq 2$ (see [14] for a nice exposition). This condition is

$$\int_0^1 \left(\frac{\operatorname{Cap}_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega, \tag{6.1.3}$$

where $\operatorname{Cap}_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$. The existence of a large solution is guaranteed for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition[18]

$$\int_a^\infty \frac{dt}{\sqrt[p]{G(t)}} < \infty \quad \text{where } \ G(t) = \int_0^t g(s)ds, \tag{6.1.4}$$

for some $a > 0$. This is an extension of the Keller-Osserman condition [8], [15], which is the above relation when $p = 2$. If for $R > diam(\Omega)$ there exists a function v which satisfies

$$\begin{aligned} -\Delta_p v + g(v) = 0 \qquad & \text{in } B_R \setminus \{0\}, \\ v = 0 \qquad & \text{on } \partial B_R, \\ \lim_{x\to 0} v(x) = \infty, & \end{aligned} \tag{6.1.5}$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \text{ in } \Omega, \tag{6.1.6}$$

is a large solution, without any assumption on the regularity of $\partial\Omega$. However the existence of a (radial) solution to problem (6.1.5) needs the fact that equation (6.1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron prove in [19] that if

$$\liminf_{|r|\to\infty} |r|^{-\frac{N(p-1)}{N-p}} \, sign(r) g(r) > 0 \quad \text{with } \ p < N, \tag{6.1.7}$$

isolated singularities of solutions of (6.1.6) are removable. Conversely, if $p-1 < q < \frac{N(p-1)}{N-p}$ with $p < N$, Friedman and Véron [5] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \tag{6.1.8}$$

with an isolated singularities. In 2003, Labutin [9] show that a necessary and sufficient condition in order the following problem be solvable

$$\begin{aligned} -\Delta u + |u|^{q-1} u = 0 \;\; \text{in } \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty, \end{aligned} \tag{6.1.9}$$

is that

$$\int_0^1 \frac{\mathrm{Cap}_{2,q'}(B_t(x)\cap\Omega^c)}{t^{N-2}}\frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega, \tag{6.1.10}$$

where $\mathrm{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$, $N \geq 3$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. We refer to [12] for other related results. Concerning the exponential case of problem (6.1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentioned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (6.1.1) be solvable in the two cases $g(u) = \mathrm{sign}(u)(e^{|u|}-1)$ and $g(u) = |u|^{q-1}u$, $q > p-1$. For $1 < p \leq N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf\left\{\sum_j h^{N-p}(B_j) : E \subset \bigcup B_j,\, diam(B_j) \leq 1\right\}, \tag{6.1.11}$$

where the B_j are balls and $h^{N-p}(B_r) = r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 2$ and $1 < p \leq N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c\cap B_r(x))}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} = +\infty \;\; \forall x \in \partial\Omega, \tag{6.1.12}$$

then there exists $u \in C^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^u - 1 = 0 \;\; in\ \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty. \end{aligned} \tag{6.1.13}$$

Clearly, when $p = N$, we have $\mathcal{H}_1^{N-p}(\{x_0\}) = 1$ for all $x_0 \in \mathbb{R}^N$ thus, (6.1.12) is true for any open domain Ω.

We also obtain a sufficient condition for the existence of a large solution in the power case expressed in terms of some $\mathrm{Cap}_{\alpha,s}$ Bessel capacity in $\mathbb{R}^N$ associated to the Besov space $B^{\alpha,s}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{Cap_{p,\frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \tag{6.1.14}$$

then, for any $p-1 < q < \frac{pq_1}{N}$ there exists $u \in C^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + u^q = 0 \quad &\text{in } \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty. & \end{aligned} \tag{6.1.15}$$

We can see that condition (6.1.12) implies (6.1.14). In view of Labutin's theorem this previous result is not optimal in the case $p = 2$, since the involved capacity is $C_{2,q_1'}$ with q_1' and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} = 0 \quad &\text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty & \end{aligned} \tag{6.1.16}$$

with $q_1 > q$.

At end we apply the previous theorem to quasilinear viscous Hamilton-Jacobi equations :

$$\begin{aligned} -\Delta_p u + a\,|\nabla u|^q + b|u|^{s-1}u = 0 \quad &\text{in } \Omega, \\ u \in C^1(\Omega),\ \lim_{\rho(x)\to 0} u(x) = \infty, & \end{aligned} \tag{6.1.17}$$

For $q_1 > p-1$ and $1 < p \leq 2$, if equation (6.1.15) admits a solution with $q = q_1$, then for any $a > 0, b > 0$ and $q \in (p-1, \frac{pq_1}{q_1+1})$, $s \in [p-1, q_1)$ there exists a positive solution to (6.1.17). Conversely, if for some $a, b > 0$, $s > p-1$ there exists a solution to equation (6.1.17) with $1 < q = p \leq 2$, then for any $q_1 > p-1$, $1 \leq q_1 \leq p$, $s_1 \geq p-1$, $a_1, b_1 > 0$ there exists a positive solution to equation (6.1.17) with parameters q_1, s_1, a_1, b_1 replacing q, s, a, b. Moreover, we also prove that the previous statement holds if for some $\gamma > 0$ there exists $u \in C(\overline{\Omega}) \cap C^1(\Omega)$, $u > 0$ in Ω satisfying

$$\begin{aligned} -\Delta_p u + u^{-\gamma} = 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{6.1.18}$$

We would like to remark that the case $p = 2$ was studied in [10]. In particular, if the boundary of Ω is smooth then (6.1.17) has a solution with $s = 1$ and $1 < q \leq 2, a > 0, b > 0$.

6.2 Morrey classes and Wolff potential estimates

In this section we assume that Ω is a bounded open subset of $\mathbb{R}^N$ and $1 < p < N$. We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (6.1.1) belongs to $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 6.2.1 *A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \leq s \leq \infty$, if there is a constant K such that*

$$\int_{\Omega \cap B_r(x)} |f| dy \leq K r^{\frac{N}{s'}} \quad \forall r > 0, \forall x \in \mathbb{R}^N. \tag{6.2.1}$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $||f||_{\mathcal{M}^s(\Omega)}$. Clearly $L^s(\Omega) \subset \mathcal{M}^s(\Omega)$.

Definition 6.2.2 *Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of nonnegative and bounded Radon measures in Ω. We define the (R-truncated) Wolff potential of μ by*

$$\mathbf{W}^R_{1,p}[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \qquad \forall x \in \mathbb{R}^N, \tag{6.2.2}$$

and the (R-truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-p}} \qquad \forall x \in \mathbb{R}^N, \tag{6.2.3}$$

where the measure is extended by 0 in Ω^c.

We recall a result proved in [6] (see also [2, Theorem 2.4]).

Theorem 6.2.3 *Let μ be a nonnegative Radon measure in $\mathbb{R}^N$. There exist positive constants C_1, C_2 depending on N, p such that*

$$\int_{2B} \exp(C_1 \mathbf{W}^R_{1,p}[\chi_B \mu]) dx \leq C_2 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $||\mathbf{M}_{p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq 1$.

For $k \geq 0$, we set $T_k(u) = sign(u) \min\{k, |u|\}$.

Definition 6.2.4 *Assume $f \in L^1_{loc}(\Omega)$. We say that a measurable function u defined in Ω is a renormalized supersolution of*

$$-\Delta_p u + f = 0 \qquad in\ \Omega \tag{6.2.4}$$

if, for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_\Omega (|\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla \varphi + f\varphi) dx \geq 0 \tag{6.2.5}$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f$ is a positive distribution in Ω.

The following result is proved in [14, Theorem 4.35].

Theorem 6.2.5 *If $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ for some $\epsilon \in (0,p)$, u is a nonnegative renormalized supersolution of* (6.2.4) *and set $\mu := -\Delta_p u + f$. Then there holds*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \geq C\mathbf{W}^{\frac{r}{4}}_{1,p}[\mu](x) \qquad \forall x \in \Omega \;\; s.t. \; B_r(x) \subset \Omega, \tag{6.2.6}$$

for some C depending only on $N, p, \varepsilon, diam(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u + f = \mu \qquad \text{in } \Omega, \tag{6.2.7}$$

where $f \in L^1(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, we have

Corollary 6.2.6 *Let $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^+_b(\Omega)$. If u is a renormalized solution to (6.2.7) and $\inf_\Omega u > -\infty$ then there exists a positive constant C depending only on $N, p, \varepsilon, diam(\Omega)$ such that*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \geq \inf_\Omega u + C\mathbf{W}^{\frac{d(x,\partial\Omega)}{4}}_{1,p}[\mu](x) \;\; \forall x \in \Omega. \tag{6.2.8}$$

The next result, proved in [2, Theorem 1.1, 1.2], is an important tool for the proof of Theorems **1** and **2**. Before presenting we introduce the notation.

Definition 6.2.7 *Let $s > 1$ and $\alpha > 0$. We denote by $C_{\alpha,s}(E)$ the Bessel capacity of Borel set $E \subset \mathbb{R}^N$,*

$$Cap_{\alpha,s}(E) = \inf\{||\phi||^s_{L^s(\mathbb{R}^N)} : \phi \in L^s_+(\mathbb{R}^N), \quad \mathbf{G}_\alpha * \phi \geq \chi_E\}$$

where χ_E is the characteristic function of E and $\mathbf{G}_\alpha$ the Bessel kernel of order α.
We say that a measure μ in Ω is absolutely continuous with respect to the capacity $Cap_{\alpha,s}$ in Ω if

$$\text{for all } \; E \subset \Omega, E \text{ Borel}, Cap_{\alpha,s}(E) = 0 \Rightarrow |\mu|(E) = 0.$$

Theorem 6.2.8 *Let $\mu \in \mathfrak{M}^b_+(\Omega)$ and $q > p-1$.*

a. *If μ is absolutely continuous with respect to the capacity $Cap_{p,\frac{q}{q+1-p}}$ in Ω, then there exists a nonnegative renormalized solution u to equation*

$$\begin{aligned} -\Delta_p u + u^q &= \mu \qquad in \;\; \Omega \\ u &= 0 \qquad on \;\; \partial\Omega, \end{aligned} \tag{6.2.9}$$

which satisfies

$$u(x) \leq C\mathbf{W}^{2diam(\Omega)}_{1,p}[\mu](x) \;\; \forall x \in \Omega. \tag{6.2.10}$$

where C is a positive constant depending on p and N.

b. *If $\exp(C\mathbf{W}^{2diam(\Omega)}_{1,p}[\mu]) \in L^1(\Omega)$ where C is the previous constant, then there exists a nonnegative renormalized solution u to equation*

$$\begin{aligned} -\Delta_p u + e^u - 1 &= \mu \qquad in \;\; \Omega \\ u &= 0 \qquad on \;\; \partial\Omega, \end{aligned} \tag{6.2.11}$$

which satisfies (6.2.10).

6.3 Estimates from below

If G is any domain in $\mathbb{R}^N$ with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (6.1.7)) there always exists a maximal solution to (6.1.6) in G. It is constructed as the limit, when $n \to \infty$, of the solutions of

$$\begin{aligned} -\Delta_p u_n + g(u_n) &= 0 \quad \text{in } G_n \\ \lim_{\rho_n(x)\to 0} u_n(x) &= \infty \\ \lim_{|x|\to\infty} u_n(x) &= 0 \quad \text{if } G_n \text{ is unbounded,} \end{aligned} \tag{6.3.1}$$

where $\{G_n\}_n$ is a sequence of smooth domains such that $G_n \subset \overline{G}_n \subset G_{n+1}$ for all n, $\{\partial G_n\}_n$ is a bounded and $\bigcup_{n=1}^{\infty} G_n = G$ and $\rho_n(x) := \mathrm{dist}(x, \partial G_n)$. Our main estimates are the following.

Theorem 6.3.1 *Let $K \subset B_{1/4}\backslash\{0\}$ be a compact set and let $U_j \in C^1(K^c)$, $j = 1, 2$, be the maximal solutions of*

$$-\Delta_p u + e^u - 1 = 0 \qquad \text{in } K^c \tag{6.3.2}$$

for U_1 and

$$-\Delta_p u + u^q = 0 \qquad \text{in } K^c \tag{6.3.3}$$

for U_2, where $p - 1 < q < \frac{pq_1}{N}$. Then there exist constants C_k, $k = 1, 2, 3, 4$, depending on N, p and q such that

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \tag{6.3.4}$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{Cap_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{6.3.5}$$

Proof. *1.* For $j \in \mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, ..., J$. By [17, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\mathrm{supp}(\mu_j) \subset K \cap S_j$, $\|\mathbf{M}_{p,1}[\mu_j]\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$c_1^{-1}\mathcal{H}_1^{N-p}(K \cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c_1 \mathcal{H}_1^{N-p}(K \cap S_j) \quad \forall j,$$

for some $c_1 = c_1(N, p)$.
Now, we will show that for $\varepsilon = \varepsilon(N, p) > 0$ small enough, there holds,

$$A := \int_{B_1} \exp\left(\varepsilon \mathbf{W}_{1,p}^1 \left[\sum_{k=3}^{J} \mu_k \right](x) \right) dx \leq c_2, \tag{6.3.6}$$

where c_2 does not depend on J.
Indeed, define $\mu_j \equiv 0$ for all $j \geq J+1$ and $j \leq 2$. We have

$$A = \sum_{j=1}^{\infty} \int_{S_j} \exp\left(\varepsilon \mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J} \mu_k\right](x)\right) dx.$$

Since for any j

$$\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J}\mu_k\right] \leq c(p)\mathbf{W}^1_{1,p}\left[\sum_{k\geq j+2}\mu_k\right] + c(p)\mathbf{W}^1_{1,p}\left[\sum_{k\leq j-2}\mu_k\right] + c(p)\sum_{k=\max\{j-1,3\}}^{j+1}\mathbf{W}^1_{1,p}[\mu_k]$$

with $c(p) = \max\{1, 5^{\frac{2-p}{p-1}}\}$ and $\exp(\sum_{i=1}^{5} a_i) \leq \sum_{i=1}^{5}\exp(5a_i)$ for all a_i. Thus,

$$A \leq \sum_{j=1}^{\infty}\int_{S_j}\exp\left(c_3\varepsilon\mathbf{W}^1_{1,p}\left[\sum_{k\geq j+2}\mu_k\right](x)\right)dx + \sum_{j=1}^{\infty}\int_{S_j}\exp\left(c_3\varepsilon\mathbf{W}^1_{1,p}\left[\sum_{k\leq j-2}\mu_k\right](x)\right)dx$$
$$+\sum_{j=1}^{\infty}\sum_{k=\max(j-1,3)}^{j+1}\int_{S_j}\exp\left(c_3\varepsilon\mathbf{W}^1_{1,p}[\mu_k](x)\right)dx := A_1 + A_2 + A_3, \text{ with } c_3 = 5c(p).$$

Estimate of A_3 : We apply Theorem 6.2.3 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} \exp\left(c_3\varepsilon\mathbf{W}^1_{1,p}[\mu_k](x)\right)dx \leq c_4 r^N_{k-1}$$

with $c_3\varepsilon \in (0, C_1]$, the constant C_1 is in Theorem 6.2.3. In particular,

$$\int_{S_j} \exp\left(c_3\varepsilon\mathbf{W}^1_{1,p}[\mu_k](x)\right)dx \leq c_4 r^N_{k-1} \text{ for } k = j-1, j, j+1,$$

which implies

$$A_3 \leq c_5 \sum_{j=1}^{+\infty} r_j^N = c_5 < \infty. \tag{6.3.7}$$

Estimate of A_1 : Since $\sum_{k\geq j+2}\mu_k(B_t(x)) = 0$ for all $x \in S_j, t \in (0, r_{j+1})$. Thus,

$$A_1 = \sum_{j=1}^{\infty}\int_{S_j}\exp\left(c_3\varepsilon\int_{r_{j+1}}^{1}\left(\frac{\sum_{k\geq j+2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right)dx$$
$$\leq \sum_{j=1}^{\infty}\exp\left(c_3\varepsilon\frac{p-1}{N-p}\left(\sum_{k\geq j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right)|S_j|.$$

Note that $\mu_k(S_k) \le \mu_k(B_{r_{k-1}}(0)) \le r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k\ge j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \le \left(\sum_{k\ge j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = (1-2^{-(N-p)})^{-\frac{1}{p-1}}.$$

Therefore

$$A_1 \le \exp\left(c_3\varepsilon\frac{p-1}{N-p}(1-2^{-(N-p)})^{-\frac{1}{p-1}}\right)|B_1| = c_6. \tag{6.3.8}$$

Estimate of A_2 : for $x\in S_j$,

$$\mathbf{W}_{1,p}^1\left[\sum_{k\le j-2}\mu_k\right](x) = \int_{r_{j-1}}^1 \left(\frac{\sum_{k\le j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} = \sum_{i=1}^{j-1}\int_{r_i}^{r_{i-1}}\left(\frac{\sum_{k\le j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}.$$

Since $r_i < t < r_{i-1}$, $\sum_{k\le i-2}\mu_k(B_t(x)) = 0, \forall i = 1,...,j-1$, thus

$$\begin{aligned}\mathbf{W}_{1,p}^1\left[\sum_{k\le j-2}\mu_k\right](x) &= \sum_{i=1}^{j-1}\int_{r_i}^{r_{i-1}}\left(\frac{\sum_{k=i-1}^{j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le \sum_{i=1}^{j-1}\int_{r_i}^{r_{i-1}}\left(\frac{\sum_{k=i-1}^{j-2}\mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\ &\le \sum_{i=1}^{j-1}\left(\sum_{k=i-1}^{j-2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \le c_7 j, \text{ with } c_7 = \left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}A_2 &\le \sum_{j=1}^{\infty}\int_{S_j}\exp(c_3c_7\varepsilon j)\,dx = \sum_{j=1}^{\infty} r_j^N\exp(c_3c_7\varepsilon j)\,|S_1|\\ &= \sum_{j=1}^{\infty}\exp\left((c_3c_7\varepsilon - N\log(2))\,j\right)|S_1| \le c_8 \quad \text{for } \ \varepsilon \le N\log(2)/(2c_3c_7).\end{aligned} \tag{6.3.9}$$

Consequently, from (6.3.8), (6.3.9) and (6.3.7), we obtain $A \le c_2 := c_6 + c_8 + c_5$ for $\varepsilon = \varepsilon(N,p)$ small enough. This implies

$$\left\|\exp\left(\frac{p}{2N}\varepsilon\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J\mu_k\right]\right)\right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1)} \le c_9\left(\int_{B_1}\exp\left(\varepsilon\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J\mu_k\right](x)\right)dx\right)^{\frac{p}{2N}} \le c_{10}, \tag{6.3.10}$$

where the constant c_{10} does not depend on J. Set $B = B_{\frac14}$. For $\varepsilon_0 = (\frac{p\varepsilon}{2NC})^{1/(p-1)}$, where C is the constant in 6.2.10, by Theorem 6.2.8 and estimate (6.3.10), there exists a nonnegative renormalized solution u to equation

$$\begin{array}{ll} -\Delta_p u + e^u - 1 = \varepsilon_0\sum_{j=3}^J\mu_j & \text{in } \ B\\ u = 0 & \text{in } \ \partial B,\end{array} \tag{6.3.11}$$

satisfying (6.2.10) with $\mu = \varepsilon_0 \sum_{j=3}^{J} \mu_j$. Thus, from Corollary 6.2.6 and estimate (6.3.10), we have

$$u(0) \geq -c_{11} + c_{12} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^{J} \mu_j \right] (0).$$

Therefore

$$\begin{aligned}
u(0) &\geq -c_{11} + c_{12} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^{J} \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= -c_{11} + c_{12} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq -c_{11} + c_{13} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&= -c_{11} + c_{13} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}.
\end{aligned}$$

From the inequality

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i,$$

we deduce that

$$\begin{aligned}
u(0) &\geq -c_{11} + c_{13} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{11} + c_{13} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{14} + c_{15} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}$$

Since U_1 is the maximal solution in K^c, u satisfies the same equation in $B \backslash K$ and $U_1 \geq u = 0$ on ∂B, it follows that U_1 dominates u in $B \backslash K$. Then $U_1(0) \geq u(0)$ and we obtain (6.3.4).

2. By [1, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $\mathrm{supp}(\mu_j) \subset K \cap S_j$ and

$$\mu_j(K \cap S_j) = \int_{\mathbb{R}^N} (G_p[\mu_j](x))^{\frac{q_1}{p-1}} dx = \mathrm{Cap}_{p, \frac{q_1}{q_1 - p + 1}}(K \cap S_j).$$

By Jensen's inequality, we have for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k \right)^s \leq \sum_{k=0}^{\infty} \theta_{k,s} a_k^s$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,s} = \begin{cases} 1 & \text{if } s \in (0,1], \\ \left(\frac{\theta+1}{\theta}\right)^{s-1}(\theta+1)^{k(s-1)} & \text{if } s > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int\limits_{B_1} \left(\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \mu_k\right](x)\right)^{q_1} dx &\le \int_{B_1}\left(\sum_{k=3}^J \theta_{k,\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right)^{q_1} dx \\
&\le \sum_{k=3}^J \theta_{k,\frac{1}{p-1}}^{q_1}\theta_{k,q_1} \int\limits_{B_1} \left(\mathbf{W}_{1,p}^1[\mu_k](x)\right)^{q_1} dx \\
&\le c_{16}\sum_{k=3}^J \theta_{k,\frac{1}{p-1}}^{q_1}\theta_{k,q_1} \int\limits_{\mathbb{R}^N} \left(G_p * \mu_k(x)\right)^{\frac{q_1}{p-1}} dx \\
&= c_{16}\sum_{k=3}^J \theta_{k,\frac{1}{p-1}}^{q_1}\theta_{k,q_1} \mathrm{Cap}_{p,\frac{q_1}{q_1-p+1}}(K\cap S_k) \\
&\le c_{17}\sum_{k=3}^J \theta_{k,\frac{1}{p-1}}^{q_1}\theta_{k,q_1} 2^{-k\left(N-\frac{pq_1}{q_1-p+1}\right)} \\
&\le c_{18},
\end{aligned}$$

for θ small enough. Here the third inequality follows from [2, Theorem 2.3] and the constant c_{18} does not depend on J. Hence,

$$\left\|\left(\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \mu_k\right]\right)^q\right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1)} \le c_{19}\left\|\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \mu_k\right]\right\|_{L^{q_1}(B_1)}^q \le c_{20}, \tag{6.3.12}$$

where c_{20} is independent of J. Take $B = B_{\frac{1}{4}}$. Since $\sum_{j=3}^J \mu_j$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{p,\frac{q}{q+1-p}}$ in B, thus by Theorem 6.2.8, there exists a nonnegative renormalized solution u to equation

$$\begin{aligned} -\Delta_p u + u^q = \textstyle\sum_{j=3}^J \mu_j \quad & \text{in } B \\ u = 0 \quad & \text{on } \partial B. \end{aligned} \tag{6.3.13}$$

satisfying (6.2.10) with $\mu = \sum_{j=3}^J \mu_j$. Thus, from Corollary 6.2.6 and estimate (6.3.12), we have

$$u(0) \ge -c_{21} + c_{22}\mathbf{W}_{1,p}^{\frac{1}{4}}\left[\sum_{j=3}^J \mu_j\right](0).$$

As above, we also get that

$$u(0) \ge -c_{23} + c_{24}\int_0^1 \left(\frac{\mathrm{Cap}_{p,\frac{q_1}{q_1-p+1}}(K\cap B_r)}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

After we also have $U_2(0) \ge u(0)$. Therefore, we obtain(6.3.5). ■

6.4 Proof of the main results

First, we prove theorem **1** in the case case $p = N$. To do this we consider the function

$$x \mapsto U(x) = U(|x|) = \log\left(\frac{N-1}{2^{N+1}}\frac{1}{R^N}\left(\frac{R}{|x|}+1\right)\right) \quad \text{in} \quad B_R(0)\backslash\{0\}.$$

One has

$$U^{'}(|x|) = \frac{1}{R+|x|} - \frac{1}{|x|} \quad \text{and} \quad U^{''}(|x|) = -\frac{1}{(R+|x|)^2} + \frac{1}{|x|^2},$$

thus, for any $0 < |x| < R$,

$$\begin{aligned}
-\Delta_N U + e^U - 1 &= -(N-1)|U^{'}(|x|)|^{N-2}\left(U^{''}(|x|) + \frac{1}{|x|}U^{'}(|x|)\right) + e^U - 1 \\
&= -\frac{(N-1)R^{N-1}}{(R+|x|)^N|x|^{N-1}} + \frac{N-1}{2^{N+1}}\frac{1}{R^N}\left(\frac{R}{|x|}+1\right) - 1 \\
&\leq -\frac{(N-1)R^{N-1}}{(2R)^N|x|^{N-1}} + \frac{N-1}{2^{N+1}}\frac{1}{R^N}\frac{2R}{|x|} \\
&\leq -1.
\end{aligned}$$

Hence, if $u \in C^1(\Omega)$ is the maximal solution of

$$-\Delta_N u + e^u - 1 = 0 \ \text{ in } \Omega \tag{6.4.1}$$

and $R = 2\mathrm{diam}(\Omega)$, then $u(x) \geq U(|x-y|)$ for any $x \in \Omega$ and $y \in \partial\Omega$. Therefore, u is a large solution and satisfies

$$u(x) \geq \log\left(\frac{N-1}{2^{N+1}}\frac{1}{R^N}\left(\frac{R}{\rho(x)}+1\right)\right) \quad \forall\, x \in \Omega.$$

Now, we prove theorem **1** in the case $p < N$ and theorem **2**. Let $u, v \in C^1(\Omega)$ be the maximal solutions of

$$\begin{aligned}
&(i) \qquad -\Delta_p u + e^u - 1 = 0 \quad \text{in } \Omega, \\
&(ii) \qquad -\Delta_p v + v^q = 0 \qquad \text{in } \Omega.
\end{aligned}$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}(z_0)}$. Let $U_1, U_2 \in C^1(K^c)$ be the maximal solutions of (6.3.2) and (6.3.3) respectively. We have $u \geq U_1$ and $v \geq U_2$ in Ω. By Theorem 6.3.1,

$$\begin{aligned}
U_1(z_0) &\geq -c_1 + c_2\int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K\cap B_r(z_0))}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} \\
&\geq -c_1 + c_2\int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K\cap B_{r-|z_0|})}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0))) \\
&\geq -c_1 + c_2\int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K\cap B_{\frac{r}{2}})}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} \\
&\geq -c_1 + c_3\int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K\cap B_r)}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}.
\end{aligned}$$

We deduce

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -c_1 + c_3 \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.$$

Similarly, we also obtain

$$\inf_{B_\delta \cap \Omega} v \geq -c_4 + c_5 \int_\delta^{1/2} \left(\frac{\mathrm{Cap}_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.$$

Therefore, u and v satisfy (6.1.13) and (6.1.15) respectively. This completes the proof.

6.5 Large solutions of quasilinear Hamilton-Jacobi equations

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 2$. In this section we use our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{gathered} -\Delta_p u + a\,|\nabla u|^q + bu^s = 0 \qquad \text{in } \Omega, \\ \lim_{\rho(x)\to 0} u(x) = \infty, \end{gathered} \tag{6.5.1}$$

where $a > 0, b > 0$ and $1 \leq q < p \leq 2$, $q > p-1, s \geq p-1$.

First we have the result of existence solutions to equation (6.5.1).

Proposition 6.5.1 *Let $a > 0, b > 0$ and $q > p-1, s \geq p-1$, $1 \leq q \leq p$ and $1 < p \leq 2$. There exists a maximal nonnegative solution $u \in C^1(\Omega)$ to equation*

$$-\Delta_p u + a\,|\nabla u|^q + bu^s = 0 \quad \text{in } \Omega \tag{6.5.2}$$

which satisfies

$$u(x) \leq c(N,p,s) b^{-\frac{1}{s-p+1}} \rho(x)^{-\frac{p}{s-p+1}} \quad \forall x \in \Omega, \tag{6.5.3}$$

if $s > p-1$,

$$u(x) \leq c(N,p,q) \left(a^{-\frac{1}{q-p+1}} \rho(x)^{-\frac{p-q}{q-p+1}} + a^{-\frac{1}{q-p+1}} b^{-\frac{1}{p-1}} \rho(x)^{-\frac{q}{(p-1)(q-p+1)}} \right) \quad \forall x \in \Omega, \tag{6.5.4}$$

if $p-1 < q < p$ and $s = p-1$, and

$$u(x) \leq c(N,p) a^{-1} b^{-\frac{1}{p-1}} \rho(x)^{-\frac{p}{p-1}} \quad \forall x \in \Omega, \tag{6.5.5}$$

if $q = p$ and $s = p-1$.

Proof. Case $s = p-1$ and $p-1 < q < p$. We consider

$$U_1(x) = U_1(|x|) = c_1 \left(\frac{R^{p'} - |x|^{p'}}{p' R^{p'-1}} \right)^{-\frac{p-q}{q-p+1}} + c_2 \in C^1(B_R(0)).$$

with $p' = \frac{p}{p-1}$ and $c_1, c_2 > 0$. We have

$$U_1'(|x|) = \frac{c_1(p-q)}{q-p+1}\frac{|x|^{p'-1}}{R^{p'-1}}\left(\frac{R^{p'}-|x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}},$$

$$U_1''(|x|) = \frac{c_1(p-q)(p'-1)}{q-p+1}\frac{|x|^{p'-2}}{R^{p'-1}}\left(\frac{R^{p'}-|x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}}$$
$$+\frac{c_1(p-q)}{(q-p+1)^2}\left(\frac{|x|^{p'-1}}{R^{p'-1}}\right)^2\left(\frac{R^{p'}-|x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{1}{q-p+1}-1},$$

and

$$A = -\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq -\Delta_p U_1 + a|\nabla U_1|^q + bc_2^{p-1}.$$

Thus, for all $x \in B_R(0)$

$$\begin{aligned}
A &\geq -(p-1)|U_1'(|x|)|^{p-2}U_1''(|x|) - \frac{N-1}{|x|}|U_1'(|x|)|^{p-2}U_1'(|x|) + a|U_1'(|x|)|^q + bc_1^{p-1} \\
&= \left(\frac{c_1(p-q)(p'-1)}{q-p+1}\right)^{p-1}\left(\frac{R^{p'}-|x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{q}{q-p+1}}\left\{-(p-1)\frac{p'-1}{p'}\left(1-\left(\frac{|x|}{R}\right)^{p'}\right)\right. \\
&\quad - \frac{1}{q-p+1}\left(\frac{|x|}{R}\right)^{p'} - \frac{N-1}{p'}\left(\frac{|x|}{R}\right)^{p'}\left(1-\left(\frac{|x|}{R}\right)^{p'}\right) \\
&\quad \left. + a\left(\frac{c_1(p-q)}{q-p+1}\right)^{q-p+1}\left(\frac{|x|}{R}\right)^{\frac{q}{q-p+1}}\right\} + bc_2^{p-1} \\
&\geq \left(\frac{c_1(p-q)(p'-1)}{q-p+1}\right)^{p-1}\left(\frac{R^{p'}-|x|^{p'}}{p'R^{p'-1}}\right)^{-\frac{q}{q-p+1}} \\
&\quad \times\left\{-\frac{N(p-1)}{p} - \frac{1}{q-p+1} + a\left(\frac{c_1(p-q)}{q-p+1}\right)^{q-p+1}\left(\frac{|x|}{R}\right)^{\frac{q}{q-p+1}}\right\} + bc_2^{p-1}.
\end{aligned}$$

Clearly, one can find $c_1 = c_2(N,p,q)a^{-\frac{1}{q-p+1}} > 0$ and $c_3 = c_3(N,p,q) > 0$ such that

$$A \geq -c_3 a^{-\frac{p-1}{q-p+1}}R^{-\frac{q}{q-p+1}} + bc_2^{p-1}.$$

Choosing $c_2 = c_3^{\frac{1}{p-1}}a^{-\frac{1}{q-p+1}}b^{-\frac{1}{p-1}}R^{-\frac{q}{(p-1)(q-p+1)}}$, we get

$$-\Delta_p U_1 + a|\nabla U_1|^q + bU_1^{p-1} \geq 0 \ \text{ in } B_R(0). \tag{6.5.6}$$

Likewise, we can verify that the function U_2 below

$$U_2(x) = c_4 a^{-1}\log\left(\frac{R^{p'}}{R^{p'}-|x|^{p'}}\right) + c_4 a^{-1}b^{-\frac{1}{p-1}}R^{-\frac{p}{p-1}}$$

belongs to $C^1_+(B_R(0))$ and satisfies

$$-\Delta_p U_2 + a|\nabla U_2|^p + bU_2^{p-1} \geq 0 \ \text{ in } B_R(0). \tag{6.5.7}$$

While, if $s > p-1$,

$$U_3(x) = c_5 b^{-\frac{1}{s-p+1}} \left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{-\frac{p}{s-p+1}}$$

belongs to $C^1(B_R(0))$ and verifies

$$-\Delta_p U_3 + bU_3^s \geq 0 \ \text{ in } B_R(0), \tag{6.5.8}$$

for some positive constants $c_4 = c_4(N,p,q)$, $c_5 = c_5(N,p,s)$ and $\beta = \beta(N,p,q) > 1$.
We emphasize the fact that with the condition $1 < p \leq 2$ and $q \geq 1$, equation (6.5.2) satisfies a comparison principle, see [16, Theorem 3.5.1, corollary 3.5.2]. Take a sequence of smooth domains Ω_n satisfying $\Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ for all n and $\bigcup\limits_{n=1}^{\infty} \Omega_n = \Omega$. For each $n,k \in \mathbb{N}^*$, there exist nonnegative solution $u_{n,k} = u \in W^{1,p}_k(\Omega_n) := W^{1,p}_0(\Omega_n) + k$ of equation (6.5.2) in Ω_n.
Since $-\Delta_p u_{k,n} \leq 0$ in Ω_n, so using the maximum principle we get $u_{n,k} \leq k$ in Ω_n for all n. Thus, by standard regularity (see [4] and [11]), $u_{n,k} \in C^{1,\alpha}(\overline{\Omega_n})$ for some $\alpha \in (0,1)$. It follows from the comparison principle and (6.5.6)-(6.5.8), that

$$u_{n,k} \leq u_{n,k+1} \qquad \text{in } \ \Omega_n$$

and (6.5.3)-(6.5.5) are satisfied with $u_{n,k}$ and Ω_n in place of u and Ω respectively. From this, we derive uniform local bounds for $\{u_{n,k}\}_k$, and by standard interior regularity (see [4]) we obtain uniform local bounds for $\{u_{n,k}\}_k$ in $C^{1,\eta}_{loc}(\Omega_n)$. It implies that the sequence $\{u_{n,k}\}_k$ is pre-compact in C^1. Therefore, up to a subsequence, $u_{n,k} \to u_n$ in $C^1(\Omega_n)$. Hence, we can verify that u_n is a solution of (6.5.2) and satisfies (6.5.3)-(6.5.5) with u_n and Ω_n replacing u and Ω and $u_n(x) \to \infty$ as $d(x, \partial\Omega_n) \to 0$.
Next, since $u_{n,k} \geq u_{n+1,k}$ in Ω_n there holds $u_n \geq u_{n+1}$ in Ω_n. In particular, $\{u_n\}$ is uniformly locally bounded in Ω. Arguing as above, we obtain $u_n \to u$ in $C^1(\Omega)$, thus u is a solution of (6.5.2) in Ω and satisfies (6.5.3)-(6.5.5). Clearly, u is the maximal solution of (6.5.2). ∎

Theorem 6.5.2 *Let $q_1 > p-1$ and $1 < p \leq 2$. Assume that equation* (6.1.15) *admits a solution with $q = q_1$. Then for any $a > 0, b > 0$ and $q \in (p-1, \frac{pq_1}{q_1+1})$, $s \in [p-1, q_1)$ equation* (6.5.2) *has a large solution satisfying* (6.5.3) *and* (6.5.4).

Proof. Assume that equation (6.1.15) admits a solution v with $q = q_1$ and set $v = \beta w^\sigma$ with $\beta > 0, \sigma \in (0,1)$, then $w > 0$ and

$$-\Delta_p w + (-\sigma+1)(p-1)\frac{|\nabla w|^p}{w} + \beta^{q_1-p+1}\sigma^{-p+1} w^{\sigma(q_1-p+1)+p-1} = 0 \text{ in } \Omega. \tag{6.5.9}$$

If we impose $\max\{\frac{s-p+1}{q_1-p+1}, \left(\frac{q}{p-q} - p + 1\right)\frac{1}{q_1-p+1}\} < \sigma < 1$, we can see that

$$(-\sigma+1)(p-1)\frac{|\nabla w|^p}{w} + \beta^{q_1-p+1}\sigma^{-p+1} w^{\sigma(q_1-p+1)+p-1} \geq a|\nabla w|^q + bw^s \ \text{ in } \ \{x : w(x) \geq M\},$$

where a positive constant M depends on p, q_1, q, s, a, b. Therefore

$$-\Delta_p w + a\,|\nabla w|^q + bw^s \leq 0 \quad \text{in} \ \ \{x : w(x) \geq M\}.$$

Now we take an open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ such that the set $\{x : w(x) \geq M\}$ contains $\Omega\backslash\overline{\Omega'}$. So w is a subsolution of $-\Delta_p u + a\,|\nabla u|^q + bu^s = 0$ in $\Omega\backslash\overline{\Omega'}$ and the same property holds with $w_\varepsilon := \varepsilon w$ for any $\varepsilon \in (0,1)$. Let u be as in Proposition 6.5.1. Set $\min\{u(x) : x \in \partial\Omega'\} = \theta_1 > 0$ and $\max\{w(x) : x \in \partial\Omega'\} = \theta_2 \geq M$. Thus $w_\varepsilon < u$ on $\partial\Omega'$ with $\varepsilon < \min\{\frac{\theta_1}{\theta_2}, 1\}$. Hence, from the construction of u in the proof of Proposition 6.5.1 and the comparison principle, we obtain $w_\varepsilon \leq u$ in $\Omega\backslash\overline{\Omega'}$. This implies the result. ∎

Remark 6.5.3 *From the proof of above Theorem, we can show that under the assumption as in Proposition 6.5.1, equation* (6.5.2) *has a large solution in* Ω *if and only if equation* (6.5.2) *has a large solution in* $\Omega\backslash K$ *for some a compact set* $K \subset \Omega$ *with smooth boundary.*

Now we deal with (6.5.1) in the case $q = p$.

Theorem 6.5.4 *Assume that equation* (6.5.2) *has a large solution in* Ω *for some* $a, b > 0$, $s > p-1$ *and* $q = p > 1$. *Then for any* $a_1, b_1 > 0$ *and* $q_1 > p-1, s_1 \geq p-1$, $1 \leq q_1 \leq p \leq 2$, *equation* (6.5.2) *also has a large solution* u *in* Ω *with parameters* a_1, b_1, q_1, s_1 *in place of* a, b, q, s *respectively, and it satisfies* (6.5.3)-(6.5.5).

Proof. For $\sigma > 0$ we set $u = v^\sigma$ thus

$$-\Delta_p v - (\sigma-1)(p-1)\frac{|\nabla v|^p}{v} + a\sigma v^{\sigma-1}\,|\nabla v|^p + b\sigma^{-p+1}v^{(s-p+1)\sigma+p-1} = 0.$$

Choose $\sigma = \frac{s_1-p+1}{s-p+1} + 2$, it is easy to see that

$$-\Delta_p v + a_1|\nabla v|^{q_1} + b_2 v^{s_1} \leq 0 \ \ \text{in} \ \ \{x : v(x) \geq M\},$$

for some a positive constant M only depending on $p, s, a, b, a_1, b_1, q_1, s_1$. Similarly as in the proof of Theorem 6.5.2, we get the result as desired. ∎

Remark 6.5.5 *If we set* $u = e^v$ *then* v *satisfies*

$$-\Delta_p v + be^{(s-p+1)v} = |\nabla v|^p\,(p-1-ae^v) \qquad in\ \Omega.$$

From this, we can construct a large solution of

$$-\Delta_p u + be^{(s-p+1)u} = 0 \qquad in\ \Omega\backslash K,$$

for any a compact set $K \subset \Omega$ *with smooth boundary such that* $v \geq ln\left(\frac{p-1}{a}\right)$ *in* $\Omega\backslash K$. *In case* $p = 2$, *It would be interesting to see what Wiener type criterion is implied by the existence as such a large solution. We conjecture that this condition must be*

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x)\cap\Omega^c)}{r^{N-2}}\frac{dr}{r} = \infty \qquad \forall x \in \partial\Omega.$$

We now consider the function

$$U_4(x) = c\left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{\frac{p}{\gamma+p-1}} \quad \text{in } B_R(0), \gamma > 0. \tag{6.5.10}$$

As in the proof of proposition 6.5.1, it is easy to check that there exist positive constants β large enough and c small enough so that inequality $\Delta_p U_4 + U_4^{-\gamma} \geq 0$ holds.
From this, we get the existence of minimal solution to equation

$$\Delta_p u + u^{-\gamma} = 0 \quad \text{in } \Omega. \tag{6.5.11}$$

Proposition 6.5.6 *Assume $\gamma > 0$. Then there exists a minimal solution $u \in C^1(\Omega)$ to equation* (6.5.11) *and it satisfies $u(x) \geq C\rho(x)^{\frac{p}{\gamma+p-1}}$ in Ω.*

We can verify that if the boundary of Ω is satisfied (6.1.3), then above minimal solution u belongs to $C(\overline{\Omega})$, vanishes on $\partial\Omega$ and it is therefore a solution to the quenching problem

$$\begin{aligned} \Delta_p u + u^{-\gamma} &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.5.12}$$

Theorem 6.5.7 *Let $\gamma > 0$. Assume that there exists a solution $u \in C(\overline{\Omega})$ to problem* (6.5.12). *Then, for any $a, b > 0$ and $q > p-1, s \geq p-1$, $1 \leq q \leq p \leq 2$, equation* (6.5.2) *admits a large solution in Ω and it satisfies* (6.5.3)-(6.5.5).

Proof. We set $u = e^{-\frac{a}{p-1}v}$, then v is a large solution of

$$-\Delta_p v + a\,|\nabla v|^p + \left(\tfrac{p-1}{a}\right)^{p-1} e^{\frac{a}{p-1}(\gamma+p-1)v} = 0 \quad \text{in } \Omega. \tag{6.5.13}$$

So

$$-\Delta_p v + a\,|\nabla v|^q + bv^s \leq 0 \quad \text{in } \{x : v(x) \geq M\}$$

for some a positive constant M only depending on p, q, s, a, b, γ. Similarly to the proof of Theorem 6.5.2, we get the result as desired. ■

Bibliographie

[1] D. R. Adams, L. I. Hedberg : *Function spaces and potential theory.* Grundlehren der Mathematischen Wissenschaften **314**, Springer-Verlag, Berlin, 1996. xii+366 pp.

[2] M.F. Bidaut-Véron, H. Nguyen Quoc, L. Véron : *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl., **102**, 315-337 (2014).

[3] G. Dal Maso, F. Murat, L. Orsina, A. Prignet : *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa **28**, 741-808 (1999).

[4] E. DiBenedetto : *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis **7**, 827-850 (1983).

[5] A. Friedman, L. Véron : *Singular Solutions of Some Quasilinear Elliptic Equations*, Arch. Rat. Mech. Anal. **96**, 259-287 (1986).

[6] P. Honzik, B. Jaye : *On the good-λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140**, 4167-4180 (2012).

[7] T. Kilpelainen, J. Malý : *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172**, 137-161 (1994).

[8] J. B. Keller : *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10**, 503-510 (1957).

[9] D. Labutin : *Wiener regularity for large solutions of nonlinear equations*, Ark. Mat. **41**, no. 2, 307-39 (2003).

[10] J.M. Lasry and P.L. Lions : *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints : 1. The Model problem*, Math. Ann. **283**, 583-630(1989).

[11] G.M. Lieberman : *Boundary regularity for solution of degenerate elliptic equations* , Nonlinear Analysis **12**, 1203-1219(1988).

[12] M. Marcus, L. Véron : *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91**, 256-295 (2009).

[13] V. Maz'ya : *On the continuity at a boundary point of solutions of quasilinear equations*, Vestnik Leningrad Univ. Math. **3**, 225-242 (1976).

[14] J. Maly, W.P. Ziemer : *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, A.M.S (1997).

[15] R. Osserman : *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7**, 1641-1647 (1957).

[16] P. Pucci, J. Serrin : *The Maximum Principle*, Progress in Nonlinear Differential Equations and Their Applications, 2007.

[17] B. O. Turesson : *Nonlinear Potential Theory and Sobolev Spaces*, Lecture Notes in Mathematics **1736**, Springer-Verlag Berlin, Heidelberg (2000).

[18] J. L. Vazquez : *An a priori interior estimate for the solution of a nonlinear problem representing weak diffusion*, Nonlinear Anal. T. M. A. **5**, 95-103 (1981).

[19] J. L. Vazquez, L. Véron : *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33**, 129-144 (1980).

[20] L. Véron : *On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data*, Math. Z. **273**, 1-2, 1-17 (2013).

[21] N. Wiener : *The Dirichlet problem*, J. Math. Phys. **3**, 127-146 (1924).

[22] W. Ziemer : *Weakly Differentiable Functions*, Graduate Texts in Mathematics **120**, Springer-Verlag, Berlin (1989).

6.6 Appendix

In this section is to establish behavior of high order gradient of the solution to equation (6.1.15) near boundary of Ω, where $\partial\Omega \in C^2$.

It is well known that if $\partial\Omega \in C^2$ then there exists $r > 0$ such that $\overline{B(y - r\vec{n}_y, r))} \cap \partial\Omega = \overline{B(y + r\vec{n}_y, r))} \cap \partial\Omega = \{y\}$ $\forall y \in \partial\Omega$, where $\vec{n}_y$ is the unique outward normal unit vector at y. Therefore, for any $x \in \Omega_r := \{x \in \Omega : \rho(x) < r\}$, there exist a unique $y \in \partial\Omega$ such that $x = y - \rho(x)\vec{n}_y$, for simplicity we write $y = Pr_{\partial\Omega}x$. We prove

Theorem 6.6.1 *Let $\partial\Omega \in C^2$ and $r > 0$ be the same as above. Then, problem* (6.1.15) *has a unique solution u which satisfies*

a. *for any $y \in \partial\Omega$ and $\beta \geq \beta_0$,*

$$C_0\left(\frac{|x - y - r\overrightarrow{n}_y|^\beta - r^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}} \leq u(x) \leq C_0\left(\frac{r^\beta - |x - y + r\overrightarrow{n}_y|^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}} \tag{6.6.1}$$

for any $x \in B(y - r\overrightarrow{n}_y, r)$, where $\vec{n}_y$ is the outward normal unit vector at y and

$$C_0 = \left(\frac{p^{p-1}(p-1)(q+1)}{(q-p+1)^p}\right)^{\frac{1}{q-p+1}}, \quad \beta_0 = \max\left\{\frac{p}{p-1}, \frac{(n-p)(q-p+1)}{p(p-1)}\right\}.$$

b. *There exists $r_0 \in (0, r)$ depending on p, q, N, Ω such that $u \in C^\infty_{loc}(\Omega_{r_0})$ where Ω_{r_0}.*

c. *Let $\vec{n}_{x_{\partial\Omega}} = (n_{x_{\partial\Omega},1}, ..., n_{x_{\partial\Omega},N})$ be the outward normal unit vector at $x_{\partial\Omega} = Pr_{\partial\Omega}x$ for all $x \in \Omega_r$. For any $m \in \mathbb{N}^*$, there exists a positive constant C depending on p, q, N, m such that $m = i_1 + i_2 + ... + i_N$*

$$\left|(\rho(x))^{\frac{p}{q-p+1}+m}\frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} - C_0\prod_{j=1}^m\left(\frac{p}{q-p+1} + j - 1\right)\prod_{k=1}^N n_{x_{\partial\Omega},k}^{i_k}\right| \leq C(\frac{\rho(x)}{r})^{\frac{1}{m+1}} \tag{6.6.2}$$

for all $x \in \Omega$ and $\rho(x) < \frac{r_0}{16}$.

Let u be a solution of problem (6.1.15). First we consider

$$U(x) = C_0\left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{-\frac{p}{q-p+1}} \quad \forall x \in B(0,R),$$

$$V(x) = C_0\left(\frac{|x|^\beta - R^\beta}{\beta R^{\beta-1}}\right)^{-\frac{p}{q-p+1}} \quad \forall x \in \mathbb{R}^N \backslash B(0,R),$$

where $R > 0$. By computing,

$$-\Delta_p U + U^q = A_1 B \text{ and } -\Delta_p V + V^q = A_2 B,$$

where

$$A_1 = \left(C_0 \frac{p}{q-p+1}\right)^{p-1}\left(\frac{R^\beta - |x|^\beta}{\beta R^{\beta-1}}\right)^{-\frac{qp}{q-p+1}},$$

$$A_2 = \left(C_0 \frac{p}{q-p+1}\right)^{p-1}\left(\frac{|x|^\beta - R^\beta}{\beta R^{\beta-1}}\right)^{-\frac{qp}{q-p+1}} \quad \text{and}$$

$$B = \left(\frac{n-p}{\beta} - \frac{p(p-1)}{q-p+1}\right)\left|\frac{x}{R}\right|^{p(\beta-1)} - \left(\frac{n-p}{\beta} + p - 1\right)\left|\frac{x}{R}\right|^{\beta(p-1)-p} + \frac{(q+1)(p-1)}{q-p+1}.$$

We see that B is decreasing with respect to $\left|\frac{x}{R}\right|$. Which implies $B \geq 0 \ \forall x \in B(0,R)$ and $B \leq 0 \ \forall x \in \mathbb{R}^N \backslash B(0,R)$. Thus,

$$-\Delta_p U + U^q \geq 0 \ \forall x \in B(0,R),$$

and

$$-\Delta_p V + V^q \leq 0 \ \forall x \in \mathbb{R}^N \backslash B(0,R).$$

So, thanks to the comparison principle we obtain (6.6.1). Hence,

$$\lim_{x \to \partial\Omega} \rho(x)^{\frac{p}{q-p+1}} u(x) = C_0,$$

and u is a unique solution of problem (6.1.15).
To prove **b.** and **c.**, we introduce the higher order divided differences.
For $h \in \mathbb{R}^N$ and $k \in \mathbb{Z}$, we set

$$\Delta_h f_k(x) = f(x + (k+1)h) - f(x + kh) \text{ for all } x \in \mathbb{R}^N.$$

By induction, we can define

$$\Delta_h^n f_k = \Delta_h\left(\Delta_h^{n-1} f_k\right),$$

for any positive integer n and

$$\Delta_{h_2}^n \Delta_{h_1}^m f_k = \Delta_{h_2}^n\left(\Delta_{h_1}^m f_k\right),$$

for any $h_1, h_2 \in \mathbb{R}^N$ and positive integers n, m. By above definition, it is not difficult to show that for any positive integers $i_1, ..., i_n$ and $h_1, ..., h_n \in \mathbb{R}^N$

$$\Delta_{h_n}^{i_n}\Delta_{h_{n-1}}^{i_{n-1}}...\Delta_{h_1}^{i_1} f(x)$$
$$= \sum_{j_n=1}^{i_n} ... \sum_{j_1=1}^{i_1} (-1)^{i_1+...+i_n+j_1+...+j_n} \left(\begin{array}{c} i_1 \\ j_1 \end{array}\right) ... \left(\begin{array}{c} i_n \\ j_n \end{array}\right) f(x + j_1h_1 + ... + j_nh_n),$$

and if $f \in C^\alpha$, $\alpha = i_1 + i_2 + ... + i_n$ then

$$\Delta_{h_n}^{i_n}\Delta_{h_{n-1}}^{i_{n-1}}...\Delta_{h_1}^{i_1} f(x)$$
$$= \int\limits_{[0,1]^\alpha} D^\alpha f(x + \sum_{k=1}^{n} (t_{1,k} + t_{2,k} + ... + t_{i_k,k})h_k) \left(h_1^{i_1}, ..., h_n^{i_n}\right) dt_{1,1}...dt_{i_1,1}...dt_{1,n}...dt_{i_n,n}.$$

Hence,

$$\sum_{j_n=1}^{i_n} ... \sum_{j_1=1}^{i_1} (-1)^{i_1+...+i_n+j_1+...+j_n} \left(\begin{array}{c} i_1 \\ j_1 \end{array}\right) ... \left(\begin{array}{c} i_n \\ j_n \end{array}\right) f(x + j_1h_1 + ... + j_nh_n)$$
$$= \int\limits_{[0,1]^\alpha} D^\alpha f(x + \sum_{k=1}^{n} (t_{1,k} + t_{2,k} + ... + t_{i_k,k})h_k) \left(h_1^{i_1}, ..., h_n^{i_n}\right) dt_{1,1}...dt_{i_1,1}...dt_{1,n}...dt_{i_n,n}.$$

In particular,

$$\Delta_{s_Ne_N}^{i_N}\Delta_{s_{N-1}e_{N-1}}^{i_{N-1}}...\Delta_{s_1e_1}^{i_1} f(x)$$
$$= \sum_{j_N=1}^{i_N} ... \sum_{j_1=1}^{i_1} (-1)^{i_1+...+i_N+j_1+...+j_N} \left(\begin{array}{c} i_1 \\ j_1 \end{array}\right) ... \left(\begin{array}{c} i_N \\ j_N \end{array}\right) f(x + j_1s_1e_1 + ... + j_Ns_Ne_N)$$
$$= \int\limits_{[0,1]^\alpha} \frac{\partial^\alpha f}{\partial x_1^{i_1}...\partial x_N^{i_N}}(x + \sum_{k=1}^{N} (t_{1,k} + ... + t_{i_k,k})s_ke_k)s_1^{i_1}...s_N^{i_N} dt_{1,1}...dt_{i_1,1}...dt_{1,n}...dt_{i_N,N} \tag{6.6.3}$$

for $\alpha = i_1 + ... + i_N$ and $s_1, s_2, ..., s_N \in \mathbb{R}$. Clearly, for any $t_1, ..., t_N \in \mathbb{R}$, there exists $(t_{1,0}, ..., t_{N,0}) \in [-t_1, i_1 - t_1] \times ... \times [-t_N, i_N - t_N]$ such that

$$\int\limits_{[0,1]^\alpha} \frac{\partial^\alpha f}{\partial x_1^{i_1}...\partial x_N^{i_N}}(x + \sum_{k=1}^{N} (t_{1,k} + t_{2,k} + ... + t_{i_k,k} - t_k)s_ke_k)s_1^{i_1}...s_N^{i_N} dt_{1,1}...dt_{i_1,1}...dt_{1,n}...dt_{i_N,N}$$
$$= \frac{\partial^\alpha f}{\partial x_1^{i_1}...\partial x_N^{i_N}}(x + \sum_{k=1}^{N} t_{k,0}s_ke_k)s_1^{i_1}...s_N^{i_N}$$

For this reason, we can find $(t_1, ..., t_N) \in [0, i_1] \times ... \times [0, i_N]$ (depending on $x, s_1, ..s_N, i_1, ..., i_N$)

such that $t_{1,0} = ... = t_{N,0} = 0$, this means

$$\frac{\partial^\alpha f(x)}{\partial x_1^{i_1}...\partial x_n^{i_n}} s_1^{i_1}...s_n^{i_n} = \Delta^{i_n}_{s_n e_n}\Delta^{i_{n-1}}_{s_{n-1}e_{n-1}}...\Delta^{i_1}_{s_1 e_1} f(x - \sum_{k=1}^{n} t_k s_k e_k)$$
$$= \int\limits_{[0,1]^\alpha} \frac{\partial^\alpha f}{\partial x_1^{i_1}...\partial x_n^{i_n}}(x + \sum_{k=1}^{n}(t_{1,k} + ... + t_{i_k,k} - t_k)s_k e_k) s_1^{i_1}...s_n^{i_n} dt_{1,1}...dt_{i_1,1}...dt_{1,n}...dt_{i_n,n} \tag{6.6.4}$$

Now we assume that $u \in C^m_{loc}(\Omega_{r_1})$ where $\Omega_{r_1} = \{x \in \Omega : \rho(x) < r_1\}$ and $r_1 \in (0, r]$. Let $x \in \Omega$ with $\rho(x) < \frac{r_1}{16}$. Using (6.6.1) where $y = Pr_{\partial\Omega}x = x_{\partial\Omega}$ and $\beta = \beta_0 + 2$

$$u_2(z) \le u(z) \le u_1(z) \ \ \forall z \in B(x_{\partial\Omega} - r\vec{n}_{x_{\partial\Omega}}, r)$$

where,

$$u_1(z) = C_0\left(\frac{r^\beta - |z - x_{\partial\Omega} + r\vec{n}_{x_{\partial\Omega}}|^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}},$$

and

$$u_2(z) = C_0\left(\frac{|z - x_{\partial\Omega} - r\vec{n}_{x_{\partial\Omega}}|^\beta - r^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}}.$$

Let $\delta \in (0, \frac{1}{2m})$ and $m = i_1 + i_2 + ... + i_N$. Using (6.6.4), we have

$$\delta^m(\rho(x))^m \frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} = \Delta^{i_N}_{\delta\rho(x)e_N}\Delta^{i_{N-1}}_{\delta\rho(x)e_{N-1}}...\Delta^{i_1}_{\delta\rho(x)e_1} u(x - \sum_{k=1}^{N} t_k\delta\rho(x)e_k)$$

for some $(t_1, ..., t_N) \in [0, i_1] \times ... \times [0, i_N]$ depending on $x, \rho(x), i_1, ..., i_N, p, q$. We can write

$$\Delta^{i_N}_{\delta\rho(x)e_N}\Delta^{i_{N-1}}_{\delta\rho e_{N-1}}...\Delta^{i_1}_{\delta\rho(x)e_1} u(x - \sum_{k=1}^{N} t_k\delta\rho(x)e_k)$$
$$= \Delta^{i_N}_{\delta\rho(x)e_N}\Delta^{i_{N-1}}_{\delta\rho(x)e_{N-1}}...\Delta^{i_1}_{\delta\rho(x)e_1} u_1(x - \sum_{k=1}^{N} t_k\delta\rho(x)e_k)$$
$$+ \Delta^{i_N}_{\delta\rho(x)e_N}\Delta^{i_{N-1}}_{\delta\rho(x)e_{N-1}}...\Delta^{i_1}_{\delta\rho(x)e_1} (u - u_1)(x - \sum_{k=1}^{N} t_k\delta\rho(x)e_k).$$

Thus

$$A - B \le \delta^m(\rho(x))^m \frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} \le A + B \tag{6.6.5}$$

where,

$$A = \Delta^{i_N}_{\delta\rho(x)e_N}\Delta^{i_{N-1}}_{\delta\rho(x)e_{N-1}}...\Delta^{i_1}_{\delta\rho(x)e_1} u_1(x - \sum_{k=1}^{N} t_k\delta\rho(x)e_k),$$

$$B=\sum_{j_N=1}^{i_N}...\sum_{j_1=1}^{i_1}\begin{pmatrix} i_1 \\ j_1\end{pmatrix}...\begin{pmatrix} i_N \\ j_N\end{pmatrix}|(u_1-u_2)\left(x+(j_1-t_1)\delta\rho(x)e_1+...+(j_N-t_N)\delta\rho(x)e_N\right)|$$

We need to estimate A and B.
Estimate B. We have,

$$\begin{aligned} x+(j_1-t_1)\delta\rho(x)e_1+...+(j_N-t_N)\delta\rho(x)e_N-x_{\partial\Omega} &= \left(-\vec{n}_{x_{\partial\Omega}}+\delta\sum_{k=1}^{N}(j_k-t_k)e_k\right)\rho(x) \\ &= \left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)\rho(x), \end{aligned}$$

where $\vec{v}_x=\delta\sum\limits_{k=1}^{N}(j_k-t_k)e_k$ with $|\vec{v}_x|\le\delta m(\le\frac{1}{2})$. We now set

$$H(t)=C_0\left(\frac{|(-1+t)\rho(x)(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x)+r\vec{n}_{x_{\partial\Omega}}|^{\beta}-|t\rho(x)(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x)+r\vec{n}_{x_{\partial\Omega}}|^{\beta}}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}}$$

for all $t\in[0,1]$ and we can write

$$(u_1-u_2)\left(x+(j_1-t_1)\delta\rho(x)e_1+...+(j_N-t_N)\delta\rho(x)e_N\right)=H(1)-H(0).$$

We will show that $|H'(t)|\le C_1\frac{1}{r}\left(\rho(x)\right)^{-\frac{p}{q-p+1}+1}$ for any $t\in[0,1]$, for some a positive constant C_1. Then,

$$|(u_1-u_2)\left(x+(j_1-t_1)\delta\rho(x)e_1+...+(j_N-t_N)\delta\rho(x)e_N\right)|\le C_1\frac{1}{r}\left(\rho(x)\right)^{-\frac{p}{q-p+1}+1}.$$

We conclude that

$$B\le 2^m C_1\frac{1}{r}(\rho(x))^{-\frac{p}{q-p+1}+1} \tag{6.6.6}$$

In fact, for $t\in[0,1]$

$$\begin{aligned} &H'(t)=-\tfrac{p}{q-p+1}C_0\left(\tfrac{\left|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}-\left|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}}{\beta r^{\beta-1}}\right)^{-\frac{q+1}{q-p+1}} \\ &\tfrac{1}{r^{\beta-1}}\Big(|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}|^{\beta-2}\left((-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right) \\ &-|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}|^{\beta-2}\left(t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right)\Big)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right) \end{aligned}$$

Since

$$\left||x|^{\beta-2}x-|y|^{\beta-2}y\right|\le(\beta-1)\,|x-y|\,(|x|+|y|)^{\beta-2}\quad x,y\in\mathbb{R}^N,$$

thus we have

$$\begin{aligned} &|H'(t)|\le\tfrac{p}{q-p+1}C_0\left(\tfrac{\left|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+v_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}-\left|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+v_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}}{\beta r^{\beta-1}}\right)^{-\frac{q+1}{q-p+1}}(\beta-1)(\rho(x))^2 \\ &\tfrac{1}{r}|-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x|^2\left(\left|(-1+t)\tfrac{\rho(x)}{r}\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+\vec{n}_{x_{\partial\Omega}}\right|+\left|t\tfrac{\rho(x)}{r}\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+\vec{n}_{x_{\partial\Omega}}\right|\right)^{\beta-2} \\ &\le\tfrac{p(\beta-1)}{q-p+1}5^{\beta-1}\tfrac{1}{r}C_0\left(\tfrac{\left|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}-\left|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^{\beta}}{\beta r^{\beta-1}}\right)^{-\frac{q+1}{q-p+1}}(\rho(x))^2 \end{aligned}$$

On the other hand,

$$\frac{\left|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^\beta-\left|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}\right|^\beta}{\beta r^{\beta-1}}$$
$$= \rho(x)\int_0^1 \left|(-s+t)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)\frac{\rho(x)}{r}+\vec{n}_{x_{\partial\Omega}}\right|^{\beta-2}\left((s-t)|-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x|^2\frac{\rho(x)}{r}+(1-\vec{n}_{x_{\partial\Omega}}\vec{v}_x)\right)ds$$

Since,

$$\left|(-s+t)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)\frac{\rho(x)}{r}+\vec{n}_{x_{\partial\Omega}}\right| \geq 1-|-s+t|\,|-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x|\,\frac{\rho(x)}{r} \geq 1-2\frac{\rho(x)}{r} \geq \frac{1}{2},$$

and

$$(s-t)|-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x|^2\frac{\rho(x)}{r}+(1-\vec{n}_{x_{\partial\Omega}}\vec{v}_x) \geq -4\frac{\rho(x)}{r}+(1-|\vec{v}_x|) \geq -4\frac{\rho(x)}{r}+\frac{1}{2} \geq \frac{1}{4}.$$

Thus,

$$\frac{|(-1+t)\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}|^\beta-|t\rho(x)\left(-\vec{n}_{x_{\partial\Omega}}+\vec{v}_x\right)+r\vec{n}_{x_{\partial\Omega}}|^\beta}{\beta r^{\beta-1}} \geq 2^{-\beta}\rho(x). \tag{6.6.7}$$

We deduce $|H'(t)| \leq C_1\frac{1}{r}\left(\rho(x)\right)^{-\frac{p}{q-p+1}+1}$ where $C_1 = \frac{p(\beta-1)}{q-p+1}5^{\beta-1}2^{\frac{(q+1)\beta}{q-p+1}}C_0$

Estimate A.

Using 6.6.4, we have

$$A = \delta^m(\rho(x))^m\frac{\partial^m u_1}{\partial x_1^{i_1}...\partial x_N^{i_N}}(x+\sum_{k=1}^N(\bar{t}_k-t_k)\delta\rho(x)e_k)$$

for some $(\bar{t}_1...,\bar{t}_N) \in [0,i_1]\times...\times[0,i_n]$.

We have

$$x+\sum_{k=1}^N(\bar{t}_k-t_k)\delta\rho(x)e_k-x_{\partial\Omega} = \left(-\vec{n}_{x_{\partial\Omega}}+\delta\sum_{k=1}^N(\bar{t}_k-t_k)e_k\right)\rho(x) = \left(-\vec{n}_{x_{\partial\Omega}}+\vec{w}_x\right)\rho(x)$$

where $\vec{w}_x = \delta\sum\limits_{k=1}^N(\bar{t}_k-t_k)e_k$ with $|\vec{w}_x| \leq \delta m(\leq\frac{1}{2})$.

It is easy to see that,

$$A = \delta^m(\rho(x))^mC_0\prod_{j=1}^m\left(\frac{p}{q-p+1}+j-1\right)\left(\frac{r^\beta-|\left(-\vec{n}_{x_{\partial\Omega}}+\vec{w}_x\right)\rho(x)+r\vec{n}_{x_{\partial\Omega}}|^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}-m}$$
$$\left|\left(-\vec{n}_{x_{\partial\Omega}}+\vec{w}_x\right)\frac{\rho(x)}{r}+\vec{n}_{x_{\partial\Omega}}\right|^{m(\beta_1-2)}\prod_{k=1}^N\left(\left(-n_{x_{\partial\Omega},k}+w_{x,k}\right)\frac{\rho(x)}{r}+n_{x_{\partial\Omega},k}\right)^{i_k}$$
$$+\frac{1}{r}\delta^m(\rho(x))^m\left(\frac{r^\beta-|\left(-\vec{n}_{x_{\partial\Omega}}+\vec{w}_x\right)\rho(x)+r\vec{n}_{x_{\partial\Omega}}|^\beta}{\beta r^{\beta-1}}\right)^{-\frac{p}{q-p+1}-m+1}Q(x),$$

where $|Q(x)| \le C$ for some a positive constant C depending on p, q, N, m.
We have

$$\frac{r^\beta - |(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\rho(x) + r\vec{n}_{x_{\partial\Omega}}|^\beta}{\beta r^\beta}$$
$$= \rho(x)\int_0^1 \left|(1-t)\,(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\frac{\rho(x)}{r} + \vec{n}_{x_{\partial\Omega}}\right|^{\beta-2}\left((1-t)\,(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\frac{\rho(x)}{r} + \vec{n}_{x_{\partial\Omega}}\right)$$
$$\times\,(\vec{n}_{x_{\partial\Omega}} - \vec{w}_x)\,dt = \rho(x)P(x),$$

where

$$P(x) = \left|(1-\bar{t})\,(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\frac{\rho(x)}{r} + \vec{n}_{x_{\partial\Omega}}\right|^{\beta_1-2}\left(-(1-\bar{t})|-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x|^2\frac{\rho(x)}{r} + 1 - \vec{n}_{x_{\partial\Omega}}\vec{w}_x\right)$$

for some $\bar{t} \in [0,1]$ and $P(x) \ge 2^{-\beta}$.
Thus,

$$A = \delta^m(\rho(x))^{-\frac{p}{q-p+1}}C_0\prod_{j=1}^m\left(\tfrac{p}{q-p+1} + j - 1\right)P(x)^{-\frac{p}{q-p+1}-m}\left|(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\tfrac{\rho(x)}{r} + \vec{n}_{x_{\partial\Omega}}\right|^{m(\beta-2)}$$
$$\prod_{k=1}^N\left((-n_{x_{\partial\Omega},k} + w_{x,k})\,\tfrac{\rho(x)}{r} + n_{x_{\partial\Omega},k}\right)^{i_k} + \tfrac{1}{r}\delta^m(\rho(x))^{-\frac{p}{q-p+1}+1}P(x)^{-\frac{p}{q-p+1}-m+1}Q(x).$$

From (6.6.5) and (6.6.6) we deduce

$$-2^mC_1\delta^{-m}\tfrac{\rho(x)}{r} + P(x)^{-\frac{p}{q-p+1}-m+1}Q(x)\tfrac{\rho(x)}{r} + T(x) \le (\rho(x))^{\frac{p}{q-p+1}+m}\frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} \le T(x)$$
$$+P(x)^{-\frac{p}{q-p+1}-m+1}Q(x)\tfrac{\rho(x)}{r} + 2^{-m}C_1\delta^{-m}\tfrac{\rho(x)}{r},$$

where

$$T(x) = C_0\prod_{j=1}^m\left(\tfrac{p}{q-p+1} + j - 1\right)P(x)^{-\frac{p}{q-p+1}-m}\left|(-\vec{n}_{x_{\partial\Omega}} + \vec{w}_x)\,\tfrac{\rho(x)}{r} + \vec{n}_{x_{\partial\Omega}}\right|^{m(\beta_1-2)}$$
$$\prod_{k=1}^N\left((-n_{x_{\partial\Omega},k} + w_{x,k})\,\tfrac{\rho(x)}{r} + n_{x_{\partial\Omega},k}\right)^{i_k}.$$

We can rewrite

$$-C_2\delta^{-m}\frac{\rho(x)}{r} + T(x) \le (\rho(x))^{\frac{p}{q-p+1}+m}\frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} \le T(x) + C_2\delta^{-m}\frac{\rho(x)}{r}$$

for some a positive constant C_2 only depending on p, q, N and m.
The remaining task is to prove that

$$\left|T(x) - C_0\prod_{j=1}^m\left(\frac{p}{q-p+1} + j - 1\right)\prod_{k=1}^N(n_{x_{\partial\Omega},k})^{i_k}\right| \le C_3\frac{\rho(x)}{r} + C_3\delta \tag{6.6.8}$$

In fact, we see that T is decomposed as the following

$$T(x) = C_0\prod_{j=1}^m\left(\frac{p}{q-p+1} + j - 1\right)\prod_{k=1}^N(n_{x_{\partial\Omega},k})^{i_k} + T_1(x) + T_2(x) + T_3(x)$$

where

$$T_1(x) = C_0 \prod_{j=1}^{m} \left(\frac{p}{q-p+1} + j - 1 \right) \left(\prod_{k=1}^{N} \left((-n_{x\partial\Omega,k} + w_{x,k}) \frac{\rho(x)}{r} + n_{x\partial\Omega,k} \right)^{i_k} - \prod_{k=1}^{N} n_{x\partial\Omega,k}^{i_k} \right),$$

$$\begin{aligned} T_2(x) = C_0 \prod_{j=1}^{m} \left(\frac{p}{q-p+1} + j - 1 \right) \left(\left| (-\vec{n}_{x\partial\Omega} + \vec{w}_x) \frac{\rho(x)}{r} + \vec{n}_{x\partial\Omega} \right|^{m(\beta-2)} - 1 \right) \\ \times \prod_{k=1}^{N} \left((-n_{x\partial\Omega,k} + w_{x,k}) \frac{\rho(x)}{r} + n_{x\partial\Omega,k} \right)^{i_k}, \end{aligned}$$

and

$$\begin{aligned} T_3(x) = C_0 \prod_{j=1}^{m} \left(\frac{p}{q-p+1} + j - 1 \right) \left(P(x)^{-\frac{p}{q-p+1} - m} - 1 \right) \\ \times \left| (-\vec{n}_{x\partial\Omega} + \vec{w}_x) \frac{\rho(x)}{R} + \vec{n}_{x\partial\Omega} \right|^{m(\beta-2)} \prod_{k=1}^{N} \left((-n_{x\partial\Omega,k} + w_{x,k}) \frac{\rho(x)}{r} + n_{x\partial\Omega,k} \right)^{i_k} \end{aligned}$$

It is obvious to see that

$$|T_1(x)| \le C_4 \frac{\rho(x)}{r}, |T_2(x)| \le C_5 \left| \left| (-\vec{n}_{x\partial\Omega} + \vec{w}_x) \frac{\rho(x)}{r} + \vec{n}_{x\partial\Omega} \right|^{m(\beta-2)} - 1 \right| \le C_6 \frac{\rho(x)}{r}$$

and

$$|T_3(x)| \le C_7 \left| P(x)^{-\frac{p}{q-p+1} - m} - 1 \right| \le C_8 \left| P(x) - 1 \right| \quad \text{since } P(x) \ge 2^{-\beta}.$$

Furthermore,

$$\begin{aligned} |P(x) - 1| &\le \left| (1-\bar{t}) \left(-\vec{n}_{x\partial\Omega} + \vec{w}_x \right) \frac{\rho(x)}{r} + \vec{n}_{x\partial\Omega} \right|^{\beta-2} (1-\bar{t}) |-\vec{n}_{x\partial\Omega} + \vec{w}_x|^2 \frac{\rho(x)}{r} \\ &\quad + \left| \left| (1-\bar{t}) \left(-\vec{n}_{x\partial\Omega} + \vec{w}_x \right) \frac{\rho(x)}{r} + \vec{n}_{x\partial\Omega} \right|^{\beta-2} - 1 \right| |1 - \vec{n}_{x\partial\Omega} \vec{w}_x| + |\vec{n}_{x\partial\Omega} \vec{w}_x| \\ &\le C_9 \frac{\rho(x)}{r} + |\vec{n}_{x\partial\Omega} \vec{w}_x| \\ &\le C_9 \frac{\rho(x)}{r} + |\vec{w}_x| \le C_9 \frac{\rho(x)}{r} + \delta m \end{aligned}$$

Thus

$$|T_3(x)| \le C_{10} \frac{\rho(x)}{r} + C_{10} \delta$$

Consequently, we get (6.6.8).
Then, we get that

$$\left| (\rho(x))^{\frac{p}{q-p+1} + m} \frac{\partial^m u(x)}{\partial x_1^{i_1} ... \partial x_N^{i_N}} - C_0 \prod_{j=1}^{m} \left(\frac{p}{q-p+1} + j - 1 \right) \prod_{k=1}^{N} (n_{x\partial\Omega,k})^{i_k} \right| \le C_{11} (\delta^{-m} \frac{\rho(x)}{r} + \delta)$$

By choosing $\delta = \frac{1}{2m}\left(\frac{\rho(x)}{r}\right)^{\frac{1}{m+1}}$, we obtain for any $x \in \Omega, \rho(x) < \frac{r_1}{16}$

$$\left|(\rho(x))^{\frac{p}{q-p+1}+m}\frac{\partial^m u(x)}{\partial x_1^{i_1}...\partial x_N^{i_N}} - C_0\prod_{j=1}^{m}\left(\frac{p}{q-p+1}+j-1\right)\prod_{k=1}^{N}(n_{x_{\partial\Omega},k})^{i_k}\right| \leq C_{12}(\frac{\rho(x)}{r})^{\frac{1}{m+1}}. \tag{6.6.9}$$

where $C_{12} = C_{12}(p,q,m,N)$. From case $m = 1$, since $u \in C^1_{loc}(\Omega)$ thus for any $x \in \Omega, \rho(x) < \frac{r}{16}$

$$\left|(\rho(x))^{\frac{p}{q-p+1}+1}\frac{\partial u}{\partial x_i} - \frac{p}{q-p+1}C_0 n_{x_{\partial\Omega},i}\right| \leq C_{12}(\frac{\rho(x)}{r})^{\frac{1}{2}}.$$

It leads to

$$\left|(\rho(x))^{\frac{p}{q-p+1}+1}|\nabla u| - \frac{p}{q-p+1}C_0\right| \leq C_{13}(\frac{\rho(x)}{r})^{\frac{1}{2}} \quad \forall x \in \Omega, \rho(x) < \frac{r}{16}$$

for some a positive constant C_{13} only depending on p, q, N.
Put $M = \max\left\{\left(\frac{2C_{13}(q-p+1)}{pC_0}\right)^2, 16\right\}$, we have

$$\frac{1}{2}\frac{p}{q-p+1}C_0(\rho(x))^{-\frac{q+1}{q-p+1}} \leq |\nabla u| \leq \frac{3}{2}\frac{p}{q-p+1}C_0(\rho(x))^{-\frac{q+1}{q-p+1}} \quad \forall x \in \Omega, \rho(x) < \frac{r}{M}$$

Therefore, by standard regularity theory, we obtain $u \in C^\infty_{loc}(\Omega_{r/M})$. Finally, from (6.6.9) with $r_1 = r/M$, we get (6.6.2).

Chapitre 7

Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain

Abstract

We obtain a necessary and a sufficient condition expressed in terms of Wiener type tests involving the parabolic $W^{2,1}_{q'}$- capacity, where $q' = \frac{q}{q-1}$, for the existence of large solutions to equation $\partial_t u - \Delta u + u^q = 0$ in non-cylindrical domain, where $q > 1$. Also, we provide a sufficient condition associated with equation $\partial_t u - \Delta u + e^u - 1 = 0$. Besides, we apply our results to equation : $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$ for $a, b > 0$, $1 < p < 2$ and $q > 1$.

7.1 Introduction

The aim of this paper is to study the problem of existence of large solutions to nonlinear parabolic equations with superlinear absorption in an *arbitrary* bounded open set $O \subset \mathbb{R}^{N+1}$, $N \geq 2$. These are solutions $u \in C^{2,1}(O)$ of equations

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u = 0 \qquad & \text{in } \; O, \\ \lim_{\delta \to 0} \inf_{O \cap Q_\delta(x,t)} u = \infty \qquad & \text{for all } \; (x,t) \in \partial_p O, \end{aligned} \tag{7.1.1}$$

and

$$\begin{aligned} \partial_t u - \Delta u + sign(u)(e^{|u|} - 1) = 0 \qquad & \text{in } \; O, \\ \lim_{\delta \to 0} \inf_{O \cap Q_\delta(x,t)} u = \infty \qquad & \text{for all } \; (x,t) \in \partial_p O, \end{aligned} \tag{7.1.2}$$

where $q > 1$ and $\partial_p O$ is the parabolic boundary of O, i.e, the set all points $X = (x,t) \in \partial O$ such that the intersection of the cylinder $Q_\delta(x,t) := B_\delta(x) \times (t - \delta^2, t)$ with O^c is not empty for any $\delta > 0$. By the maximal principle for parabolic equations we can assume that all solutions of (7.1.1) and (7.1.2) are positive. Henceforth we consider only positive solutions of the preceding equations.
In [22], we studied the existence and the uniqueness of solution of general equations in a cylindrical domain,

$$\begin{aligned} \partial_t u - \Delta u + f(u) = 0 \qquad & \text{in } \; \Omega \times (0,\infty), \\ u = \infty \qquad & \text{in } \; \partial_p \left(\Omega \times (0,\infty) \right), \end{aligned} \tag{7.1.3}$$

where Ω is a bounded open set in $\mathbb{R}^N$ and f is a continuous real-valued function, nondecreasing on $\mathbb{R}$ such that $f(0) \geq 0$ and $f(a) > 0$ for some $a > 0$. In order to obtain the existence of a maximal solution of $\partial_t u - \Delta u + f(u) = 0$ in $\Omega \times (0,\infty)$ there is need to assume

$$\begin{aligned} (i) \qquad & \int_a^\infty \left(\int_0^s f(\tau) d\tau \right)^{-\frac{1}{2}} ds < \infty, \\ (ii) \qquad & \int_a^\infty (f(s))^{-1} \, ds < \infty. \end{aligned} \tag{7.1.4}$$

Condition (i), due to Keller and Osserman, is a necessary and sufficient for the existence of a maximal solution to

$$-\Delta u + f(u) = 0 \;\; \text{in } \Omega. \tag{7.1.5}$$

Condition (ii) is a necessary and sufficient for the existence of a maximal solution of the differential equation

$$\varphi' + f(\varphi) = 0 \qquad \text{in } (0,\infty), \tag{7.1.6}$$

and this solution tends to ∞ at 0. In [22], it is shown that if for any $m \in \mathbb{R}$ there exists $L = L(m) > 0$ such that

$$\text{for any } x, y \geq m \Rightarrow f(x+y) \geq f(x) + f(y) - L,$$

and if (7.1.5) has a large solution, then (7.1.3) admits a solution.

It is not alway true that the maximal solution to (7.1.5) is a large solution. However, if f satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s)ds < \infty \quad \text{if } N \geq 3,$$

or

$$\inf\left\{a \geq 0 : \int_0^\infty f(s)e^{-as}ds < \infty \right\} < \infty \ \text{ if } N = 2,$$

then (7.1.5) has a large solution for any bounded domain Ω, see [16].

When $f(u) = u^q$, $q > 1$ and $N \geq 3$, the first above condition is satisfied if and only if $q < q_c := \frac{N}{N-2}$, this is called *the sub-critical case*. When $q \geq q_c$, a necessary and sufficient condition for the existence of a large solution to

$$-\Delta u + u^q = 0 \ \text{ in } \Omega; \tag{7.1.7}$$

is expressed in term of a Wiener-type test,

$$\int_0^1 \frac{\mathrm{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } \ x \in \partial\Omega. \tag{7.1.8}$$

In the case $q = 2$ it is obtained by probabilistic methods involving the Brownian snake by Dhersin and Le Gall [5], also see [13, 14]; this method can be extended for $1 < q \leq 2$ by using ideas from [7, 8]. In the general case the result is proved by Labutin, by using purely analytic methods [12]. Here, $q' = \frac{q}{q-1}$ and $\mathrm{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$.

In [19] we obtain sufficient conditions when $f(u) = e^u - 1$, involving the Hausdorff $\mathcal{H}_1^{N-2}-$capacity in $\mathbb{R}^N$, namely,

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } \ x \in \partial\Omega. \tag{7.1.9}$$

We refer to [17] for investigation of the initial trace theory of (7.1.3). In [9], Evans and Gariepy establish a Wiener criterion for the regularity of a boundary point (in the sense of potential theory) for the heat operator $L = \partial_t - \Delta$ in an arbitrary bounded set of $\mathbb{R}^{N+1}$. We denote by $\mathfrak{M}(\mathbb{R}^{N+1})$ the set of Radon measures in $\mathbb{R}^{N+1}$ and, for any compact set $K \subset \mathbb{R}^{N+1}$, by $\mathfrak{M}_K(\mathbb{R}^{N+1})$ the subset of $\mathfrak{M}(\mathbb{R}^{N+1})$ of measures with support in K. Their positive cones are respectively denoted by $\mathfrak{M}^+(\mathbb{R}^{N+1})$ and $\mathfrak{M}_K^+(\mathbb{R}^{N+1})$. The capacity used in this criterion is the thermal capacity defined by

$$\mathrm{Cap}_{\mathbb{H}}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_K^+(\mathbb{R}^{N+1}), \mathbb{H} * \mu \leq 1\},$$

for any $K \subset \mathbb{R}^{N+1}$ compact, where $\mathbb{H}$ is the heat kernel in $\mathbb{R}^{N+1}$. It coincides with the parabolic Bessel $\mathcal{G}_1-$capacity $\mathrm{Cap}_{\mathcal{G}_1,2}$,

$$\mathrm{Cap}_{\mathcal{G}_1,2}(K) = \sup\left\{\int_{\mathbb{R}^{N+1}} |f|^2 dxdt : f \in L_+^2(\mathbb{R}^{N+1}),\ \mathcal{G}_1 * f \geq \chi_K\right\},$$

here $\mathcal{G}_1$ is the parabolic Bessel kernel of first order, see [20, Remark 4.12]. Garofalo and Lanconelli [10] extend this result to the parabolic operator $L = \partial_t - \operatorname{div}(A(x,t)\nabla)$, where $A(x,t) = (a_{i,j}(x,t))$, $i,j = 1,2,...,N$ is a real, symmetric, matrix-valued function on $\mathbb{R}^{N+1}$ with C^∞ entries for which there holds

$$C^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j \leq C|\xi|^2 \quad \forall (x,t) \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^N,$$

for some constant $C > 0$.

Less is known concerning the equation

$$\partial_t u - \Delta u + f(u) = 0 \tag{7.1.10}$$

in a bounded open set $O \subset$ of $\mathbb{R}^{N+1}$, where f is a continuous function in $\mathbb{R}$, Gariepy and Ziemer [11, 23] prove that if there are $(x_0,t_0) \in \partial_p O$, $l \in \mathbb{R}$ and a weak solution $u \in W^{1,2}(O) \cap L^\infty(O)$ of (7.1.10) such that $\eta(-l-\varepsilon+u)^+, \eta(l-\varepsilon-u)^+ \in W_0^{1,2}(O)$ for any $\varepsilon > 0$ and $\eta \in C_c^\infty(B_r(x_0) \times (-r^2+t_0, r^2+t_0))$ for some $r > 0$ and if

$$\int_0^1 \frac{\mathrm{Cap}_{\mathbb{H}}\left(O^c \cap \left(B_\rho(x_0) \times (t_0 - \frac{9}{4}\alpha\rho^2, t_0 - \frac{5}{4}\alpha\rho^2)\right)\right)}{\rho^N}\frac{d\rho}{\rho} = \infty \text{ for some } \alpha > 0$$

then $\lim_{(x,t)\to(x_0,t_0)} u(x,t) = l$. This result is not easy to use because it is not clear whether (7.1.10) has a weak solution $u \in W^{1,2}(O)$. In this article we show that (7.1.10) admits a maximal solution $u \in C^{2,1}(O)$ in an arbitrary bounded open set O, by approximation by dyadic parabolic cubes from inside O, provided that f is as in (7.1.3) and satisfies (7.1.4).

Our main purpose of this article is to extend the result of Labutin [12] to nonlinear parabolic equation (7.1.1). Namely, we give a necessary and a sufficient condition for the existence of solutions to (7.1.1) in a bounded non-cylindrical domain $O \subset \mathbb{R}^{N+1}$, expressed in terms of a Wiener test based upon the parabolic $W^{2,1}_{q'}$-capacity in $\mathbb{R}^{N+1}$. We also give a sufficient condition associated (7.1.2) where the parabolic $W^{2,1}_{q'}$-capacity is replaced the parabolic Hausdorff $\mathcal{PH}^N_\rho$-capacity. These capacities are defined as follows : if $K \subset \mathbb{R}^{N+1}$ is compact set, we set

$$\mathrm{Cap}_{2,1,q'}(K) = \inf\{||\varphi||^{q'}_{W^{2,1}_{q'}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},$$

where

$$||\varphi||_{W^{2,1}_{q'}(\mathbb{R}^{N+1})} = ||\varphi||_{L^{q'}(\mathbb{R}^{N+1})} + ||\frac{\partial\varphi}{\partial t}||_{L^{q'}(\mathbb{R}^{N+1})} + ||\nabla\varphi||_{L^{q'}(\mathbb{R}^{N+1})} + \sum_{i,j}||\frac{\partial^2\varphi}{\partial x_i \partial x_j}||_{L^{q'}(\mathbb{R}^{N+1})}.$$

and for Suslin set $E \subset \mathbb{R}^{N+1}$,

$$\mathrm{Cap}_{2,1,q'}(E) = \sup\{\mathrm{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\}.$$

This capacity has been used in order to obtain potential theory estimates that are most helpful for studying quasilinear parabolic equations (see e.g. [3, 4, 20]). Thanks to a result due to Richard and Bagby [2], the capacities $\mathrm{Cap}_{2,1,p}$ and $\mathrm{Cap}_{\mathcal{G}_2,p}$ are equivalent in the sense that, for any Suslin set $K \subset \mathbb{R}^{N+1}$, there holds

$$C^{-1}\mathrm{Cap}_{2,1,q'}(K) \leq \mathrm{Cap}_{\mathcal{G}_2,q'}(K) \leq C\mathrm{Cap}_{2,1,p}(K),$$

for some $C = C(N,q)$, where $\mathrm{Cap}_{\mathcal{G}_2,q'}$ is the parabolic Bessel $\mathcal{G}_2-$capacity, see [20].
For $E \subset \mathbb{R}^{N+1}$, we define $\mathcal{PH}^N_\rho(E)$ by

$$\mathcal{PH}^N_\rho(E) = \inf\left\{\sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2),\ r_j \leq \rho\right\}.$$

It is easy to see that, for $0 < \sigma \leq \rho$ and $E \subset \mathbb{R}^{N+1}$, there holds

$$\mathcal{PH}^N_\rho(E) \leq \mathcal{PH}^N_\sigma(E) \leq C(N)\left(\frac{\rho}{\sigma}\right)^2 \mathcal{PH}^N_\rho(E). \tag{7.1.11}$$

With these notations, we can state the two main results of this paper.

Theorem 7.1.1 *Let $N \geq 2$ and $q \geq q_* := \frac{N+2}{N}$. Then*
(i) The equation

$$\partial_t u - \Delta u + u^q = 0 \ \textit{in}\ \ O \tag{7.1.12}$$

admits a large solution if there holds

$$\sum_{k=1}^{\infty} \frac{Cap_{2,1,q'}\left(O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right)}{r_k^N} = \infty, \tag{7.1.13}$$

for any $(x,t) \in \partial_p O$, where $r_k = 4^{-k}$, and $N \geq 3$ when $q = q_*$.
(ii) If equation (7.1.12) admits a large solution, then

$$\int_0^1 \frac{Cap_{2,1,q'}(O^c \cap Q_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} = \infty, \tag{7.1.14}$$

for any $(x,t) \in \partial_p O$, where $Q_\rho(x,t) = B_\rho(x) \times (t - \rho^2, t)$.

Theorem 7.1.2 *Let $N \geq 2$. The equation*

$$\partial_t u - \Delta u + e^u - 1 = 0 \ \textit{in}\ \ O \tag{7.1.15}$$

admits a large solution if there holds

$$\sum_{k=1}^{\infty} \frac{\mathcal{PH}^N_1\left(O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right)}{r_k^N} = \infty, \tag{7.1.16}$$

for any $(x,t) \in \partial_p O$, with $r_k = 4^{-k}$.

From properties of the $W^{2,1}_{q'}-$capacity and the $\mathcal{PH}^N_1-$capacity, relation (7.1.13) holds if

$$\sum_{k=1}^{\infty} r_k^{-N} \left|O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right|^{1-\frac{2q'}{N+2}} = \infty \text{ when } q > q_*,$$

and

$$\sum_{k=1}^{\infty} r_k^{-N} \log_+ \left(\left|O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right|^{-1}\right)^{-\frac{N}{2}} = \infty \text{ when } q = q_*.$$

Similarly, identity (7.1.16) is verified if

$$\sum_{k=1}^{\infty} r_k^{-N} \left|O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right|^{\frac{N}{N+2}} = \infty.$$

Therefore, when $O = \{(x,t) \in \mathbb{R}^{N+1} : |x|^2 + \frac{|t|^2}{\lambda} < 1\}$ for some $\lambda > 0$, we see that $\partial O = \partial_p O$, (7.1.14) holds for any $(x,t) \in \partial_p O$, (7.1.13) and (7.1.16) hold for any $(x,t) \in \partial_p O \backslash \{(0, \sqrt{\lambda})\}$. However, (7.1.13) and (7.1.16) are also true at $(x,t) = (0, \sqrt{\lambda})$ if $\lambda > 2272^2$ and not true if $\lambda < 2272^2$.

As a consequence of Theorem 7.1.1 we derive a sufficient condition for the existence of large solution of some viscous Hamilton-Jacobi parabolic equations.

Theorem 7.1.3 *Let $q_1 > 1$. If there exists a large solution $v \in C^{2,1}(O)$ of*

$$\partial_t v - \Delta v + v^{q_1} = 0 \quad \text{in } O,$$

then, for any $a, b > 0$, $1 < q < q_1$ and $1 < p < \frac{2q_1}{q_1+1}$, problem

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad & \text{in } O, \\ u = \infty \quad & \text{on } \partial_p O, \end{aligned} \tag{7.1.17}$$

admits a solution $u \in C^{2,1}(O)$ which satisfies

$$u(x,t) \geq C \min\left\{a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}}\right\} (v(x,t))^{\frac{1}{\alpha}},$$

for all $(x,t) \in O$ where $R > 0$ is such that $O \subset \tilde{Q}_R(x_0,t_0)$, $C = C(N,p,q,q_1) > 0$ and $\alpha = \max\left\{\frac{2(p-1)}{(q_1-1)(2-p)}, \frac{q-1}{q_1-1}\right\} \in (0,1)$.

7.2 Preliminaries

Throughout the paper, we denote $Q_\rho(x,t) = B_\rho(x) \times (t - \rho^2, t]$ and $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t - \rho^2, t + \rho^2)$ for $(x,t) \in \mathbb{R}^{N+1}$, $\rho > 0$ and $r_k = 4^{-k}$ for all $k \in \mathbb{Z}$. We also denote $A \lesssim (\gtrsim) B$ if $A \leq (\geq) CB$ for some C depending on some structural constants, $A \asymp B$ if $A \lesssim B \lesssim A$.

Definition 7.2.1 *Let $R \in (0,\infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. We define $R-$truncated Riesz parabolic potential $\mathbb{I}_2$ of μ by*

$$\mathbb{I}_2^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \quad \text{for all } \ (x,t) \in \mathbb{R}^{N+1},$$

and the $R-$truncated fractional maximal parabolic potential $\mathbb{M}_2$ of μ by

$$\mathbb{M}_2^R[\mu](x,t) = \sup_{0<\rho<R} \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N} \quad \text{for all } \ (x,t) \in \mathbb{R}^{N+1}.$$

We recall two results in [20].

Theorem 7.2.2 *Let $q > 1, R > 0$ and K be a compact set in $\mathbb{R}^{N+1}$. There exists $\mu := \mu_K \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with compact support in K such that*

$$\mu(K) \asymp Cap_{2,1,q'}(K) \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^{2R}[\mu]\right)^q dxdt$$

where the constants of equivalence depend on N, q and R. The measure μ_K is called the capacitary measure of K.

Theorem 7.2.3 *For any $R > 0$, there exist positive constants C_1, C_2 such that for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $||\mathbb{M}_2^R[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \le 1$, there holds*

$$\fint_Q exp(C_1\mathbb{I}_2^R[\chi_Q\mu])dxdt \le C_2,$$

for all $Q = \tilde{Q}_r(y,s) \subset \mathbb{R}^{N+1}$, $r > 0$, where χ_Q is the indicator function of Q.

Frostman's Lemma in [21, Th. 3.4.27] is at the basis of the dual definition of Hausdorff capacities with doubling weight. It is easy to see that it is valid for the parabolic Hausdorff $\mathcal{PH}_\rho^N-$capacity version. As a consequence we have

Theorem 7.2.4 *There holds*

$$\sup\left\{\mu(K) : \mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), supp(\mu) \subset K, ||\mathbb{M}_2^\rho[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \le 1\right\} \asymp \mathcal{PH}_\rho^N(K)$$

for any compact set $K \subset \mathbb{R}^{N+1}$ and $\rho > 0$, where equivalent constant depends on N

For our purpose, we need the some results about the behavior of the capacity with respect to dilations.

Proposition 7.2.5 *Let $K \subset \overline{\tilde{Q}_{100}(0,0)}$ be a compact set and $1 < p < \frac{N+2}{2}$. Then*

$$Cap_{2,1,p}(K) \gtrsim |K|^{1-\frac{2p}{N+2}}, Cap_{2,1,\frac{N+2}{2}}(K) \gtrsim \left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right)\right)^{-\frac{N}{2}}, \qquad (7.2.1)$$

and

$$Cap_{2,1,p}(K_\rho) \asymp \rho^{N+2-2p} Cap_{2,1,p}(K), \tag{7.2.2}$$

$$\frac{1}{Cap_{2,1,\frac{N+2}{2}}(K_\rho)} \asymp \frac{1}{Cap_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2} \tag{7.2.3}$$

for any $0 < \rho < 1$*, where* $K_\rho = \{(\rho x, \rho^2 t) : (x,t) \in K\}$.

Proposition 7.2.6 *Let* $K \subset \overline{\tilde{Q}_1(0,0)}$ *be a compact set and* $1 < p \leq \frac{N+2}{2}$*. Then, there exists a function* $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))$*,* $0 \leq \varphi \leq 1$ *and* $\varphi|_D = 1$ *for some open set* $D \supset K$ *such that*

$$\int_{\mathbb{R}^{N+1}} \left(|D^2\varphi|^p + |\nabla\varphi|^p + |\varphi|^p + |\partial_t\varphi|^p\right) dxdt \lesssim Cap_{2,1,p}(K). \tag{7.2.4}$$

We will give proofs of the above two propositions in the Appendix.
It is well know that there exists a semigroup $e^{t\Delta}$ corresponding to equation

$$\begin{array}{rll} \partial_t u - \Delta u = \mu & \text{in} & \tilde{Q}_R(0,0), \\ u = 0 & \text{on} & \partial_p \tilde{Q}_R(0,0), \end{array} \tag{7.2.5}$$

with $\mu \in C^\infty(\tilde{Q}_R(0,0))$, i.e, we can write a solution u of (7.2.5) as follows

$$u(x,t) = \int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

We denote by $\mathbb{H}$ the heat kernel :

$$\mathbb{H}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

We have

$$|u(x,t)| \leq (\mathbb{H} * \mu)(x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

In [20, Proof of Proposition 4.8] we show that

$$|(\mathbb{H} * \mu)|(x,t) \leq C_1(N)\mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

Here μ is extended by 0 in $(\tilde{Q}_R(0,0))^c$. Thus,

$$\left|\int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds\right| \leq C_1(N)\mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0). \tag{7.2.6}$$

Moreover, we also prove in [20], that if $\mu \geq 0$ then for $(x,t) \in \tilde{Q}_R(0,0)$ and $B_\rho(x) \subset B_R(0)$,

$$\int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds \geq C_2(N)\sum_{k=0}^\infty \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N}, \tag{7.2.7}$$

with $\rho_k = 4^{-k}\rho$.
It is easy to see that estimates (7.2.6) and (7.2.7) also holds for any bounded Radon measure μ in $\tilde{Q}_R(0,0)$. The following result is proved in [3] and [18], and also in [20] in a more general framework.

Theorem 7.2.7 *Let $q > 1$, $R > 0$ and μ be bounded Radon measure in $\tilde{Q}_R(0,0)$.*

(i) If μ is absolutely continuous with respect to $Cap_{2,1,q'}$ in $\tilde{Q}_R(0,0)$, then there exists a unique weak solution u to equation

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u = \mu \quad & \text{in } \tilde{Q}_R(0,0), \\ u = 0 \quad & \text{on } \partial_p \tilde{Q}_R(0,0). \end{aligned}$$

(ii) If $\exp\left(C_1(N)\mathbb{I}_2^{2R}[|\mu|]\right) \in L^1(\tilde{Q}_R(0,0))$ then there exists a unique weak solution v to equation

$$\begin{aligned} \partial_t v - \Delta v + sign(v)(e^{|v|} - 1) = \mu \quad & \text{in } \tilde{Q}_R(0,0), \\ v = 0 \quad & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned}$$

where the constant $C_1(N)$ is the one of inequality (7.2.6).

From estimates (7.2.6) and (7.2.7) and using comparison principle we get the estimates from below of the solutions u and v obtained in Theorem 7.2.7.

Proposition 7.2.8 *If $\mu \geq 0$ then the functions u and v of the previous theorem are non-negative and satisfy*

$$u(x,t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)^{q+1}\mathbb{I}_2^{2R}\left[\left(\mathbb{I}_2^{2R}[\mu]\right)^q\right](x,t) \tag{7.2.8}$$

and

$$v(x,t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)\mathbb{I}_2^{2R}\left[\exp\left(C_1(N)\mathbb{I}_2^{2R}[\mu]\right) - 1\right](x,t). \tag{7.2.9}$$

for any $(x,t) \in \tilde{Q}_R(0,0)$ and $B_\rho(x) \subset B_R(0)$ and $\rho_k = 4^{-k}\rho$.

7.3 Maximal solutions

In this section we assume that O is an arbitrary non-cylindrical and bounded open set in $\mathbb{R}^{N+1}$ and $q > 1$. We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0 \tag{7.3.1}$$

in O. We also get analogous result where u^q is replaced by $e^u - 1$.
It is easy to see that if u satisfies (7.3.1) in $\tilde{Q}_r(0,0)$ ($Q_r(0,0)$) then $u_a(x,t) = a^{-2/(q-1)}u(ax, a^2t)$ satisfies (7.3.1) in $\tilde{Q}_{r/a}(0,0)$ ($Q_{r/a}(0,0)$) for any $a > 0$.
If $X = (x,t) \in O$, the parabolic distance from X to the parabolic boundary $\partial_p O$ of O is defined by

$$d(X, \partial_p O) = \inf_{\substack{(y,s)\in\partial_p O \\ s \leq t}} \max\{|x-y|, (t-s)^{\frac{1}{2}}\}.$$

It is easy to see that there exists $C = C(N, q) > 0$ such that the function V defined by

$$V(x,t) = C\left((\rho^2+t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2-|x|^2}{\rho}\right)^{-\frac{2}{q-1}}\right) \quad \text{in } B_\rho(0)\times(-\rho^2,0)$$

satisfies

$$\partial_t V - \Delta V + V^q \geq 0 \quad \text{in } \; B_\rho(0)\times(-\rho^2,0). \tag{7.3.2}$$

Proposition 7.3.1 *There exists a maximal solution $u \in C^{2,1}(O)$ of (7.3.1) and it satisfies*

$$u(x,t) \leq C(d((x,t),\partial_p O))^{-\frac{2}{q-1}} \quad \textit{for all } (x,t)\in O \tag{7.3.3}$$

for some $C = C(N, q)$.

Proof. Let $\mathcal{D}_k$, $k \in \mathbb{Z}$ be the collection of all the dyadic parabolic cubes (abridged p-cubes) of the form

$$\{(x_1,...,x_N,t) : m_j 2^{-k} \leq x_j \leq (m_j+1)2^{-k}, j=1,...,N, m_{N+1}4^{-k} \leq t \leq (m_{N+1}+1)4^{-k}\}$$

where $m_j \in \mathbb{Z}$. The following properties hold,

a. for each integer k, $\mathcal{D}_k$ is a partition of $\mathbb{R}^{N+1}$ and all p-cubes in $\mathcal{D}_k$ have the same sidelengths.

b. if the interiors of two p-cubes Q in $\mathcal{D}_{k_1}$ and P in $\mathcal{D}_{k_2}$, denoted $\overset{\circ}{Q}, \overset{\circ}{P}$, have nonempty intersection then either Q is contained in R or Q contains R.

c. Each Q in $\mathcal{D}_k$ is union of 2^{N+2} p-cubes in $\mathcal{D}_{k+1}$ with disjoint interiors.

Let $k_0 \in \mathbb{N}$ be such that $Q \subset O$ for some $Q \in \mathcal{D}_{k_0}$. Set $O_k = \bigcup_{\substack{Q\in\mathcal{D}_k\\ Q\subset O}} Q \quad \forall k \geq k_0$, we have $O_k \subset O_{k+1}$ and $O = \bigcup_{k\geq k_0} O_k = \bigcup_{k\geq k_0} \overset{\circ}{O}_k$. More precisely, there exist real numbers $a_1, a_2, ..., a_{n(k)}$ and open sets $\Omega_1, \Omega_2, .., \Omega_{n(k)}$ in $\mathbb{R}^N$ such that

$$a_i < a_i + 4^{-k} \leq a_{i+1} < a_{i+1} + 4^k \quad \text{for} \quad i = 1, ..., n(k)-1$$

and

$$\overset{\circ}{O}_k = \bigcup_{i=1}^{n(k)-1}\left(\Omega_i \times (a_i, a_i+4^{-k}]\right)\bigcup\left(\Omega_{n(k)}\times(a_{n(k)}, a_{n(k)}+4^{-k})\right).$$

For $k \geq k_0$, we claim that there exists a solution $u_k \in C^{2,1}(\overset{\circ}{O}_k)$ to problem

$$\begin{aligned} \partial_t u_k - \Delta u_k + u_k^q = 0 \qquad & \text{in } \; \overset{\circ}{O}_k, \\ u_k(x,t) \to \infty \quad & \text{as } d((x,t),\partial_p \overset{\circ}{O}_k) \to 0. \end{aligned} \tag{7.3.4}$$

Indeed, by [6, 15] for $m > 0$, one can find nonnegative solutions $v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k}]) \cap C(\overline{\Omega}_i \times [a_i, a_i + 4^{-k}])$ for $i = 1, .., n(k)$ to equations

$$\begin{aligned} \partial_t v_1 - \Delta v_1 + v_1^q = 0 \qquad & \text{in } \; \Omega_1 \times (a_1, a_1+4^{-k}), \\ v_1(x,t) = m \qquad & \text{on } \; \partial\Omega_1 \times (a_1, a_1+4^{-k}), \\ v_1(x,t_1) = m \qquad & \text{in } \; \Omega_1, \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i - \Delta v_i + v_i^q = 0 \qquad & \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x,t) = m \qquad & \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \end{aligned}$$

$$v_i(x,a_i) = \begin{cases} m \ \text{ in } \ \Omega_i & \text{if } \ a_i > a_{i-1} + 4^{-k}, \\ m\chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k})\chi_{\Omega_{i-1}}(x) & \text{otherwise .} \end{cases}$$

Clearly,

$$u_{k,m} = v_i \ \text{ in } \Omega_i \times (a_i, a_i + 4^{-k}] \quad \text{for } \ i = 1, ..., n(k)$$

is a solution in $C^{2,1}(\overset{\circ}{O}_k) \cap C(O_k)$ to equation

$$\begin{cases} \partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q = 0 \text{ in } \ \overset{\circ}{O}_k, \\ u_{k,m} = m \text{ on } \ \partial_p \overset{\circ}{O}_k. \end{cases}$$

Moreover, for $(x,t) \in \overset{\circ}{O}_k$, we see that $B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t) \subset \overset{\circ}{O}_k$ where $d = d((x,t), \partial_p \overset{\circ}{O}_k)$. From (7.3.2), we verify that

$$U(y,s) := V(y-x, s-t) = C\left((\rho^2 + s - t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x-y|^2}{\rho} \right)^{-\frac{2}{q-1}} \right)$$

with $\rho = d/2$, satisfies

$$\partial_t U - \Delta U + U^q \geq 0 \quad \text{in } \ B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t). \tag{7.3.5}$$

Applying the comparison principle we get

$$u_{k,m}(y,s) \leq U(y,s) \ \text{ in } \ B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t],$$

which implies

$$u_{k,m}(x,t) \leq C\left(d((x,t), \partial_p \overset{\circ}{O}_k) \right)^{-\frac{2}{q-1}} \quad \text{for all } \ (x,t) \in \overset{\circ}{O}_k. \tag{7.3.6}$$

From this, we also obtain uniform local bounds for $\{u_{k,m}\}_m$. By standard regularity theory see [6, 15], $\{u_{k,m}\}_m$ is uniformly locally bounded in $C^{2,1}$. Hence, up to a subsequence, $u_{k,m} \to u_k$ $C^{1,0}_{\text{loc}}(\overset{\circ}{O}_k)$. Passing the limit, we derive that u_k is a weak solution of (7.3.4) in $\overset{\circ}{O}_k$, which satisfies $u_k(x,t) \to \infty$ as $d((x,t), \partial_p \overset{\circ}{O}_k) \to 0$ and

$$u_k(x,t) \leq C\left(d((x,t), \partial_p \overset{\circ}{O}_k) \right)^{-\frac{2}{q-1}} \quad \text{for all } \ (x,t) \in \overset{\circ}{O}_k.$$

Let $m > 0$ and $k \geq k_0$. Since $u_{k+1,m} \leq m$ in O_k and $O_k \subset O_{k+1}$, it follows by the comparison principle applied to $u_{k+1,m}$ and $u_{k,m}$ in the sub-domains $\Omega_1 \times (a_1, a_1 + 4^{-k})$, $\Omega_2 \times (a_2, a_2 + 4^{-k})$,..., $\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})$ of $\overset{\circ}{O}_k$ to obtain at end that $u_{k+1,m} \leq u_{k,m}$ in $\overset{\circ}{O}_k$, and thus $u_{k+1} \leq u_k$ in $\overset{\circ}{O}_k$. In particular, $\{u_k\}_k$ is uniformly locally bounded in L^∞_{loc}. We use the same compactness property as above to obtain that $u_k \to u$ where u is a solution of (7.3.1) and satisfies (7.3.3). By construction u is the maximal solution. ∎

Remark 7.3.2 *Let $R \geq 2r \geq 2$, K be a compact subset in $\overline{\tilde{Q}_r(0,0)}$. Arguing as one can easily it is clear that there exists a maximal solution of*

$$\begin{aligned} \partial_t u - \Delta u + u^q = 0 \quad & \text{in } \tilde{Q}_R(0,0)\backslash K, \\ u = 0 \quad & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned} \tag{7.3.7}$$

which satisfies

$$u(x,t) \leq C(d((x,t), \partial_p(\tilde{Q}_R(0,0)\backslash K))^{-\frac{2}{q-1}} \quad \forall\, (x,t) \in \tilde{Q}_R(0,0)\backslash K, \tag{7.3.8}$$

for some $C = C(N,q)$. Furthermore, assume $K_1, K_2,,, K_m$ are compact subsets in $\overline{\tilde{Q}_r(0,0)}$ and $K = K_1 \cup ... \cup K_m$. Let $u, u_1, ..., u_m$ be the maximal solutions of (7.3.7) in $\tilde{Q}_R(0,0)\backslash K$, $\tilde{Q}_R(0,0)\backslash K_1$, $\tilde{Q}_R(0,0)\backslash K_2,,, \tilde{Q}_R(0,0)\backslash K_m$, respectively, then

$$u \leq \sum_{j=1}^{m} u_j \quad \text{in } \tilde{Q}_R(0,0)\backslash K. \tag{7.3.9}$$

Remark 7.3.3 *If the equation (7.3.1) admits a large solution for some $q > 1$ then for any $1 < q_1 < q$, equation*

$$\partial_t u - \Delta u + u^{q_1} = 0 \text{ in } O \tag{7.3.10}$$

admits also a large solution.
Indeed, assume that u is a large solution of (7.3.1) and v is the maximal solution of (7.3.10). Take $R > 0$ such that $O \subset B_R(0) \times (-R^2, R^2)$, then the function V defined by

$$V(x,t) = (q-1)^{-\frac{1}{q-1}}(2R^2 + t)^{-\frac{1}{q-1}},$$

satisfies (7.3.1). It follows for all $(x,t) \in O$

$$u(x,t) \geq \inf_{(y,s)\in O} V(x,t) \geq (q-1)^{-\frac{1}{q-1}} R^{-\frac{2}{q-1}} =: a_0.$$

Thus, $\tilde{u} = a_0^{\frac{q-q_1}{q_1-1}} u$ is a subsolution of (7.3.10). Therefore $v \geq a_0^{\frac{q-q_1}{q_1-1}} u$ in O, thus v is a large solution.

Remark 7.3.4 (Sub-critical case) *Assume that $1 < q < q_*$. One easily see that the function*

$$U(x,t) = \frac{C}{t^{\frac{1}{q-1}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0} \tag{7.3.11}$$

is a subsolution of (7.3.1) in $\mathbb{R}^{N+1}\backslash\{(0,0)\}$, where $C = \left(\frac{2}{q-1} - \frac{N}{2}\right)^{\frac{1}{q-1}}$.
Therefore, the maximal solutions u of (7.3.1) in O verify

$$u(x,t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{t>s}, \tag{7.3.12}$$

for all $(x,t) \in O$ *and* $(y,s) \in O^c$.
If for any $(x,t) \in \partial_p O$ *there exist* $\varepsilon \in (0,1)$ *and a decreasing sequence* $\{\delta_n\} \subset (0,1)$ *converging to* 0 *as* $n \to \infty$ *such that* $\big(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)\big) \cap O^c \neq \emptyset$ *for any* $n \in \mathbb{N}$, *then* u *is a large solution. For proving this, we need to show that* $\lim_{\rho\to 0} \inf_{O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))} u = \infty$. *Let* $0 < \rho < \sqrt{\frac{\varepsilon}{2}}\delta_1$, *and* $n \in \mathbb{N}$ *such that* $\sqrt{\frac{\varepsilon}{2}}\delta_{n+1} \leq \rho < \sqrt{\frac{\varepsilon}{2}}\delta_n$.
Since $\big(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)\big) \cap O^c \neq \emptyset$, *there is* $(x_n, t_n) \in O^c$ *such that* $|x_n - x| < \delta_n$ *and* $-\delta_n^2 + t < t_n < -\varepsilon\delta_n^2 + t$. *So if* $(y,s) \in O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))$ *then* $|y - x_n| < (\sqrt{\varepsilon} + 1)\delta_n$ *and* $\frac{\varepsilon}{2}\delta_n^2 < s - t_n < (\varepsilon + 1)\delta_n^2$. *Hence, thanks to* (7.3.12) *we have for any* $(y,s) \in O \cap (B_\rho(x) \times (-\rho^2 + t, \rho^2 + t))$

$$u(y,s) \geq C\frac{1}{(s-t_n)^{\frac{1}{q-1}}} e^{-\frac{|y-x_n|^2}{4(s-t_n)}} \geq C(\varepsilon+1)^{-\frac{1}{q-1}} e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}} \delta_n^{-\frac{2}{q-1}},$$

which implies

$$\inf_{O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))} u \geq C(\varepsilon+1)^{-\frac{1}{q-1}} e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}} \delta_n^{-\frac{2}{q-1}} \to \infty \quad \textit{as} \quad \rho \to 0.$$

Remark 7.3.5 *Note that if* $u \in C^{2,1}(O)$ *is a solution of* (7.3.1) *for some* $q > 1$ *then, for* $a, b > 0$ *and* $1 < p \leq 2$, $v = b^{-\frac{1}{q-1}} u$ *is a super-solution of*

$$\partial_t v - \Delta v + a|\nabla v|^p + bv^q = 0 \quad \textit{in} \quad O. \tag{7.3.13}$$

Thus, we can apply the argument of the previous proof, with equation (7.3.1) *replaced by* (7.3.13), *and deduce that there exists a maximal solution* $v \in C^{2,1}(O)$ *of* (7.3.13) *satisfying*

$$v(x,t) \leq Cb^{-\frac{1}{q-1}} (d((x,t), \partial_p O))^{-\frac{2}{q-1}} \quad \textit{for all } (x,t) \in O.$$

Furthermore, if $1 < q < q_*$, $q = \frac{2p}{p+1}$, $a, b > 0$ *then the function* U *in Remark 7.3.4 is a subsolution of* (7.3.13) *in* $\mathbb{R}^{N+1}\backslash\{(0,0)\}$, *for some* $C = C(N,p,q,a,b)$. *Therefore, we conclude that every maximal solution of* $v \in C^{2,1}(O)$ *of* (7.3.13) *satisfy*

$$v(x,t) \geq C\frac{1}{(t-s)^{\frac{1}{q-1}}} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{t>s} \tag{7.3.14}$$

for all $(x,t) \in O$ *and* $(y,s) \in \partial_p O$.
As in Remark 7.3.4, if for any $(x,t) \in \partial_p O$ *there exist* $\varepsilon \in (0,1)$ *and a decreasing sequence* $\{\delta_n\} \subset (0,1)$ *converging to* 0 *as* $n \to \infty$ *such that* $\big(B_{\delta_n}(x) \times (-\delta_n^2 + t, -\varepsilon\delta_n^2 + t)\big) \cap O^c \neq \emptyset$ *for any* $n \in \mathbb{N}$, *then* v *is a large solution.*

Next, we consider the following equation

$$\partial_t u - \Delta u + e^u - 1 = 0. \tag{7.3.15}$$

It is easy to see that the two functions

$$V_1(t) = -\log\left(\frac{t+\rho^2}{1+\rho^2}\right) \quad \text{and} \quad V_2(x) = C - 2\log\left(\frac{\rho^2 - |x|^2}{\rho}\right)$$

satisfy

$$V_1' + e^{V_1} - 1 \geq 0 \qquad \text{in } \ (-\rho^2, 0]$$

and

$$-\Delta V_2 + e^{V_2} - 1 \geq 0 \qquad \text{in } \ B_\rho(0)$$

for some $C = C(N)$. Using $e^a + e^b \leq e^{a+b} - 1$ for $a, b \geq 0$, we obtain that $V_1 + V_2$ is a supersolution of equation (7.3.15) in $B_\rho(0) \times (-\rho^2, 0]$. By the same argument as in Proposition 7.3.1 and the estimate of the above supersolution, we obtain

Proposition 7.3.6 *There exists a maximal solution $u \in C^{2,1}(O)$ of*

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } \ O, \tag{7.3.16}$$

and it satisfies

$$u(x,t) \leq C - \log\left(\frac{(d((x,t), \partial_p O))^3}{4 + (d((x,t), \partial_p O))^2}\right) \quad \text{for all } (x,t) \in O, \tag{7.3.17}$$

for some $C = C(N)$.

The next three propositions will be useful to prove Theorem 7.1.1-(ii).

Proposition 7.3.7 *Let $K \subset \overline{\tilde{Q}_1(0,0)}$ be a compact set and $q > 1$, $R \geq 100$. Let u be a solution of (7.3.7) in $\tilde{Q}_R(0,0)\backslash K$ and φ as in Proposition 7.2.6 with $p = q'$. Set $\xi = (1-\varphi)^{2q'}$. Then,*

$$\int_{\tilde{Q}_R(0,0)} u\left(|\Delta\xi| + |\nabla\xi| + |\partial_t\xi|\right) dxdt \lesssim Cap_{2,1,q'}(K), \tag{7.3.18}$$

$$u(x,t) \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad \text{for any } \ (x,t) \in \tilde{Q}_{R/5}(0,0)\backslash\tilde{Q}_2(0,0), \tag{7.3.19}$$

and

$$\int_{\tilde{Q}_2(0,0)} u\xi dxdt \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}}, \tag{7.3.20}$$

where the constants in above inequalities depend only on N, q.

Proof. *Step 1.* We claim that

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \mathrm{Cap}_{2,1,q'}(K). \tag{7.3.21}$$

Actually, using by parts integration and the Green formula, one has

$$\begin{aligned}\int_{\tilde{Q}_R(0,0)} u^q\xi dxdt &= -\int_{\tilde{Q}_R(0,0)} \partial_t u\xi dxdt + \int_{\tilde{Q}_R(0,0)} (\Delta u)\xi dxdt \\ &= \int_{\tilde{Q}_R(0,0)} u\partial_t\xi dxdt + \int_{\tilde{Q}_R(0,0)} u\Delta\xi dxdt + \int_{-R^2}^{R^2}\int_{\partial B_R(0)} \left(\xi\frac{\partial u}{\partial\nu} - u\frac{\partial\xi}{\partial\nu}\right) dSdt\end{aligned}$$

where ν is the outer normal unit vector on $\partial B_R(0)$. Clearly,

$$\frac{\partial u}{\partial \nu} \le 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R(0).$$

Thus,

$$\begin{aligned}
&\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \le \int_{\tilde{Q}_R(0,0)} u|\partial_t \xi| dxdt + \int_{\tilde{Q}_R(0,0)} u|\Delta \xi| dxdt \\
&\le 2q' \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1}|\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-2}|\nabla \varphi|^2 dxdt \\
&\qquad + 2q' \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1}|\Delta \varphi| dxdt \\
&\le 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\nabla \varphi|^2 dxdt \\
&\qquad + 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\Delta \varphi| dxdt. \qquad (7.3.22)
\end{aligned}$$

In the last inequality, we have used the fact that $(1-\phi)^{2q'-1} \le (1-\phi)^{2q'-2} = \xi^{1/q}$. Hence, by Hölder's inequality,

$$\begin{aligned}
\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} |\partial_t \varphi|^{q'} dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt \\
+ \int_{\tilde{Q}_R(0,0)} |\Delta \varphi|^{q'} dxdt.
\end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt &\lesssim ||\varphi||^{q'}_{L^\infty(\tilde{Q}_R(0,0))} \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt \\
&\lesssim \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt.
\end{aligned}$$

Hence, we find

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} (|\partial_t \varphi|^{q'} + |D^2 \varphi|^{q'}) dxdt,$$

and derive (7.3.21) from (7.2.4). In view of (7.3.22), we also obtain

$$\int_{\tilde{Q}_R(0,0)} u(|\Delta \xi| + |\partial_t \xi|) dxdt \lesssim \mathrm{Cap}_{2,1,q'}(K)$$

and

$$\int_{\tilde{Q}_R(0,0)} u|\nabla \xi| dxdt \lesssim \mathrm{Cap}_{2,1,q'}(K),$$

since

$$\begin{aligned}\int_{\tilde{Q}_R(0,0)} u|\nabla\xi|dxdt &= 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{(2q'-1)/2q'}|\nabla\varphi|dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\nabla\varphi|dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} u^q\xi dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla\varphi|^{q'} dxdt.\end{aligned}$$

It yields (7.3.18).
Step 2. Relation (7.3.19) holds. Let η be a cut off function on $\tilde{Q}_{R/4}(0,0)$ with respect to $\tilde{Q}_{R/3}(0,0)$ such that $|\partial_t\eta| + |D^2\eta| \lesssim R^{-2}$ and $|\nabla\eta| \lesssim R^{-1}$. We have

$$\partial_t(\eta\xi u) - \Delta(\eta\xi u) = F \in C_c(\tilde{Q}_{R/3}(0,0)).$$

Hence, we can write

$$(\eta\xi u)(x,t) = \int_{\mathbb{R}^N}\int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y,s)dsdy \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

Now, we fix $(x,t) \in \tilde{Q}_{R/5}(0,0)\backslash\tilde{Q}_2(0,0)$. Since $\mathrm{supp}\{|\nabla\eta|\} \cap \mathrm{supp}\{|\nabla\xi|\} = \emptyset$ and

$$\begin{aligned}F &= \eta\xi\left(\partial_t u - \Delta u\right) - 2\left(\eta\nabla\xi + \xi\nabla\eta\right)\nabla u + \left(\xi\partial_t\eta + \eta\partial_t\xi - 2\nabla\eta\nabla\xi - \Delta\eta\xi - \eta\Delta\xi\right)u \\ &\leq -2\left(\eta\nabla\xi + \xi\nabla\eta\right)\nabla u + \left(\xi\partial_t\eta + \eta\partial_t\xi - \xi\Delta\eta - \eta\Delta\xi\right)u,\end{aligned}$$

there holds

$$\begin{aligned}u(x,t) = (\eta\xi u)(x,t) &\leq -2\int_{\mathbb{R}^N}\int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\nabla\xi + \xi\nabla\eta\right)\nabla u dsdy \\ &+ \int_{\mathbb{R}^N}\int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\partial_t\xi - \eta\Delta\xi\right) u dsdy \\ &+ \int_{\mathbb{R}^N}\int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\partial_t\eta\xi - \xi\Delta\eta\right) u dsdy. \\ &= I_1 + I_2 + I_3.\end{aligned}$$

By parts integration

$$\begin{aligned}I_1 &= 2(4\pi)^{-N/2}\int_{-\infty}^t\int_{\mathbb{R}^N} \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\nabla\xi + \xi\nabla\eta\right) u dyds \\ &+ 2(4\pi)^{-N/2}\int_{-\infty}^t\int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\xi\Delta\eta + \eta\Delta\xi\right) u\, dyds.\end{aligned}$$

Note that

$$\frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \lesssim \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N},$$

$$\left|\frac{(x-y)}{2(t-s)^{(N+2)/2}}e^{-\frac{|x-y|^2}{4(t-s)}}\right| \lesssim \left(\max\{|x-y|,|t-s|^{1/2}\}\right)^{-N-1},$$

and

$$\begin{aligned}
&\max\{|x-y|,|t-s|^{1/2}\} \gtrsim 1 \quad \forall (y,s)\in \operatorname{supp}\{|D^\alpha \xi|\}\cup \operatorname{supp}\{|\partial_t \xi|\},\\
&\max\{|x-y|,|t-s|^{1/2}\} \gtrsim R \quad \forall (y,s)\in \operatorname{supp}\{|D^\alpha \eta|\}\cup \operatorname{supp}\{|\partial_t \eta|\} \quad \forall |\alpha|\geq 1.
\end{aligned}$$

We deduce

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|,|t-s|^{1/2}\}\right)^{-N-1}(\eta|\nabla\xi|+\xi|\nabla\eta|)u\,dyds\\
&\quad+\int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|,|t-s|^{1/2}\}\right)^{-N}(\xi|\Delta\eta|+\eta|\Delta\xi|)\,u\,dyds\\
&\lesssim \int_{\mathbb{R}^{N+1}}(|\nabla\xi|+|\Delta\xi|)u\,dyds+\int_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)}(R^{-N-1}|\nabla\eta|+R^{-N}|\Delta\eta|)u\,dyds\\
&\lesssim \int_{\mathbb{R}^{N+1}}(|\nabla\xi|+|\Delta\xi|)u\,dyds+\sup_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} u,
\end{aligned}$$

$$\begin{aligned}
I_2 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|,|t-s|^{1/2}\}\right)^{-N}(|\partial_t\xi|+|\Delta\xi|)u\,dyds\\
&\lesssim \int_{\mathbb{R}^{N+1}}(|\partial_t\xi|+|\Delta\xi|)u\,dyds,
\end{aligned}$$

and

$$\begin{aligned}
I_3 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|,|t-s|^{1/2}\}\right)^{-N}(|\partial_t\eta|+|\Delta\eta|)u\,dyds\\
&\lesssim \int_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} R^{-N}(|\partial_t\eta|+|\Delta\eta|)u\,dyds\\
&\lesssim \sup_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} u.
\end{aligned}$$

Hence,

$$u(x,t)\leq I_1+I_2+I_3 \lesssim \int_{\mathbb{R}^{N+1}}(|\partial_t\xi|+|\nabla\xi|+|\Delta\xi|)u\,dyds+\sup_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} u.$$

Combining this with (7.3.18) and (7.3.8), we obtain (7.3.19).
Step 3. End of the proof. Let θ be a cut off function on $\tilde{Q}_3(0,0)$ with respect to $\tilde{Q}_4(0,0)$. As above, we have for any $(x,t)\in\mathbb{R}^{N+1}$

$$\begin{aligned}
(\theta\xi u)(x,t) &\lesssim \int_{\mathbb{R}^{N+1}}(\max\{|x-y|,|t-s|^{1/2}\})^{-N-1}(\theta|\nabla\xi|+\xi|\nabla\theta|)u\,dyds\\
&\quad+\int_{\mathbb{R}^{N+1}}(\max\{|x-y|,|t-s|^{1/2}\})^{-N}(\theta|\Delta\xi|+\xi|\Delta\theta|)u\,dyds\\
&\quad+\int_{\mathbb{R}^{N+1}}(\max\{|x-y|,|t-s|^{1/2}\})^{-N}(\theta|\partial_t\xi|+\theta|\Delta\xi|)u\,dyds\\
&\quad+\int_{\mathbb{R}^{N+1}}(\max\{|x-y|,|t-s|^{1/2}\})^{-N}(\xi|\partial_t\theta|+\xi|\Delta\theta|)u\,dyds.
\end{aligned}$$

Hence, by Fubini theorem,

$$\begin{aligned}\int_{\tilde{Q}_2(0,0)} \eta u dx dt &= \int_{\tilde{Q}_2(0,0)} \theta \eta u dx dt \\ &\lesssim A \int_{\mathbb{R}^{N+1}} (\theta|\nabla\xi| + \xi|\nabla\theta| + \theta|\Delta\xi| + \xi|\Delta\theta| + \theta|\partial_t\xi| + \xi|\partial_t\theta|)\, u\, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t\xi| + |\nabla\xi| + |\Delta\xi|) u\, dyds + \sup_{\tilde{Q}_4(0,0)\setminus\tilde{Q}_3(0,0)} u\end{aligned}$$

where

$$A = \sup_{(y,s)\in\tilde{Q}_4(0,0)} \int_{\tilde{Q}_2(0,0)} ((\max\{|x-y|, |t-s|^{1/2}\})^{-N} + (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1}) dx dt.$$

Therefore we obtain (7.3.20) from (7.3.18) and (7.3.19). ■

Proposition 7.3.8 *Let $K \subset \{(x,t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\}$ be a compact set, $0 < \varepsilon < 1$ and u be the maximal solution of (7.3.7) in $\tilde{Q}_R(0,0)\setminus K$ with $R \geq 100$. Then*

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=-2}^{j_\varepsilon - 2} \frac{Cap_{2,1,q'}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if} \quad q > q_*, \tag{7.3.23}$$

and

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=0}^{j_\varepsilon} \frac{Cap_{2,1,q'}(K_j)}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if} \quad q = q_*, \tag{7.3.24}$$

where $\rho_j = 2^{-j}$, $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x,t) \in K \cap \tilde{Q}_{\rho_{j-2}}(0,0)\}$ and $j_\varepsilon \in \mathbb{N}$ is such that $\rho_{j_\varepsilon} \leq \varepsilon < \rho_{j_\varepsilon - 1}$.

Proof. For $j \in N$, we define $S_j = \{x : \rho_j \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-1}\}$.
Fix any $1 \leq j \leq j_\varepsilon$. We cover S_j by $L = L(N) \in \mathbb{N}^*$ closed cylinders

$$\overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}, \quad k = 1, ..., L(N)$$

where $(x_{k,j}, t_{k,j}) \in S_j$.
For $k = 1, ..., L(N)$, let $u_j, u_{k,j}$ be the maximal solutions of (7.3.7) where K is replaced by $K \cap S_j$ and $K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}$, respectively. Clearly the function $\tilde{u}_{k,j}$ defined by

$$\tilde{u}_{k,j}(x,t) = \rho_{j+3}^{\frac{2}{q-1}} u_{k,j}(\rho_{j+3}x + x_{k,j}, \rho_{j+3}^2 t + t_{k,j})$$

is the maximal solution of (7.3.7) when $(K_{k,j}, \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2))$ is replacing $(K, \tilde{Q}_R(0,0))$, with

$$K_{k,j} = \{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y,s) \in -(x_{k,j}, t_{k,j}) + K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}\} \subset \overline{\tilde{Q}_1(0,0)}.$$

Let $\overline{u}_{k,j}$ be the maximal solution of (7.3.7) with $(K, \tilde{Q}_R(0,0))$ replaced by $(K_{k,j}, \tilde{Q}_{2R/\rho_{j+3}}(0,0))$. Since $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2) \subset \tilde{Q}_{2R/\rho_{j+3}}(0,0)$, then, by the comparison principle as in the proof of Proposition 7.3.1 we get $\tilde{u}_{k,j} \leq \overline{u}_{k,j}$ in $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2)\backslash K_{k,j}$ and thus

$$\tilde{u}_{k,j}(x,t) \lesssim \mathrm{Cap}_{2,1,q'}(K_{k,j}) + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

for any $(x,t) \in \left(\tilde{Q}_{2R/(5\rho_{j+3})}(0,0) \cap \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2)\right) \backslash \tilde{Q}_2(0,0) = D$. Fix $(x_0,t_0) \in \tilde{Q}_{\varepsilon/4}(0,0)$. Clearly, $((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}) \in D$, hence

$$\begin{aligned} u_{k,j}(x_0,t_0) &= \rho_{j+3}^{-\frac{2}{q-1}} \tilde{u}_{k,j}((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \\ &\lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_{k,j})}{\rho_j^{\frac{2}{q-1}}} + R^{-\frac{2}{q-1}}. \end{aligned}$$

Therefore, using (7.3.9) in Remark 7.3.2 and the fact that

$$\mathrm{Cap}_{2,1,q'}(K_{k,j}) = \mathrm{Cap}_{2,1,q'}(K_{k,j} + (x_{k,j}/\rho_{j+3}, t_{k,j}/\rho_{j+3}^2)) \leq \mathrm{Cap}_{2,1,q'}(K_j),$$

we derive

$$\begin{aligned} u(x_0,t_0) \leq \sum_{j=1}^{j_\varepsilon} u_j(x_0,t_0) &\leq \sum_{j=1}^{j_\varepsilon} \sum_{k=1}^{L(N)} u_{k,j}(x_0,t_0) \\ &\lesssim \sum_{j=0}^{j_\varepsilon} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_j^{\frac{2}{q-1}}} + j_\varepsilon R^{-\frac{2}{q-1}}, \end{aligned}$$

which yields (7.3.24). If $q > q_*$, then by (7.2.2) in Proposition 7.2.5, we have

$$\mathrm{Cap}_{2,1,q'}(K_j) \lesssim \rho_{j+3}^{-N-2+2q'} \mathrm{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_{j-2}}(0,0)),$$

which implies (7.3.23). ∎

Proposition 7.3.9 *Let K, u, ξ be as in Proposition 7.3.7. For any compact set K_0 in $\tilde{Q}_1(0,0)$ with positive measure $|K_0|$, there exists $\varepsilon = \varepsilon(N,q,|K_0|) > 0$ such that*

$$Cap_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \int_{\tilde{Q}_2(0,0)} u\xi dxdt,$$

where the constant in the inequality $\lesssim$ depends on K_0. In particular,

$$Cap_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}}. \tag{7.3.25}$$

Proof. It is enough to prove that there exists $\varepsilon > 0$ such that

$$\mathrm{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_1| \geq 1/2|K_0| \tag{7.3.26}$$

where $K_1 = \{(x,t) \in K_0 : \xi(x,t) \geq 1/2\}$. By (7.2.1) in Proposition 7.2.5, we have the following estimates

$$|K_0 \backslash K_1|^{1-\frac{2q'}{N+2}} \lesssim \mathrm{Cap}_{2,1,q'}(K_0 \backslash K_1)$$

if $q > q_*$, and

$$\left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K_0 \backslash K_1|}\right)\right)^{-\frac{N}{2}} \lesssim \mathrm{Cap}_{2,1,q'}(K_0 \backslash K_1)$$

if $q = q_*$. On the other hand,

$$\begin{aligned} \mathrm{Cap}_{2,1,q'}(K_0 \backslash K_1) &= \mathrm{Cap}_{2,1,q'}(\{K_0 : \varphi > 1 - (1/2)^{1/(2q')}\}) \\ &\leq (1 - (1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^{N+1}} \left(|D^2\varphi|^{q'} + |\nabla \varphi|^{q'} + |\varphi|^{q'} + |\partial_t \varphi|^{q'}\right) dxdt \\ &\lesssim \mathrm{Cap}_{2,1,q'}(K) \end{aligned}$$

where φ is in Proposition 7.3.7. Henceforth, one can find $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that

$$\mathrm{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0 \backslash K_1| \leq 1/2 \ |K_0|.$$

This implies (7.3.26). ∎

7.4 Large solutions

In the first part of this section, we prove theorem 7.1.1-(ii), then we prove theorems 7.1.1-(i) and 7.1.2, at end we consider a parabolic viscous Hamilton-Jacobi equation.

7.4.1 Proof of Theorem 7.1.1-(ii)

Let $R_0 \geq 4$ such that $O \subset\subset \tilde{Q}_{R_0}(0,0)$. Assume that the equation (7.1.12) has a large solution u. Take any $(x,t) \in \partial_p O$. We will to prove that (7.1.14) holds. We can assume $(x,t) = (0,0)$. Set $K = \tilde{Q}_{2R_0}(0,0) \backslash O$ and define

$$\begin{aligned} T_j &= \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\}, \\ \tilde{T}_j &= \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-2}, t \leq 0\}. \end{aligned}$$

Here $\rho_j = 2^{-j}$. For $j \geq 3$, let u_1, u_2, u_3, u_4 be the maximal solutions of (7.3.7) when K is replaced by $K \cap \overline{Q_{\rho_{j+3}}(0,0)}$, $K \cap \tilde{T}_j$, $\left(K \cap \overline{Q_1(0,0)}\right) \backslash Q_{\rho_{j-2}}(0,0)$ and $K \backslash Q_1(0,0)$ respectively and $R \geq 100 R_0$. From (7.3.9) in Remark 7.3.2, we can assert that

$$u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in} \ \ O \cap \{(x,t) \in \mathbb{R}^{N+1} : t \leq 0\}.$$

Thus,

$$\inf_{T_j} u \leq ||u_1||_{L^\infty(T_j)} + ||u_3||_{L^\infty(T_j)} + ||u_4||_{L^\infty(T_j)} + \inf_{T_j} u_2. \tag{7.4.1}$$

Case 1 : $q > q_*$. By (7.3.8) in Remark 7.3.2,

$$||u_4||_{L^\infty(T_j)} \lesssim 1. \tag{7.4.2}$$

By (7.3.23) in Proposition 7.3.8,

$$||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}}. \tag{7.4.3}$$

Since $(x,t) \mapsto \overline{u}_1(x,t) = \rho_{j+3}^{2/(q-1)} u_1(\rho_{j+3}x, \rho_{j+3}^2 t)$ is the maximal solution of (7.3.7) when $(K, \tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y,s) \in K \cap \overline{Q_{\rho_{j+3}}(0,0)}\}, \tilde{Q}_{R/\rho_{j+3}}(0,0))$, we derive, thanks to (7.3.19) in Proposition 7.3.7 and (7.2.2) in Proposition 7.2.5,

$$||\overline{u}_1||_{L^\infty(T_{-3})} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^{N+2-2q'}} + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

from which follows

$$||u_1||_{L^\infty(T_j)} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}. \tag{7.4.4}$$

Since, $(x,t) \mapsto \overline{u}_2(x,t) = \rho_{j-2}^{2/(q-1)} u_2(\rho_{j-2}x, \rho_{j-2}^2 t)$ is the maximal solution of (7.3.7) when the couple $(K, \tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j-2}, s/\rho_{j-2}^2) : (y,s) \in K \cap \tilde{T}_j\}, \tilde{Q}_{R/\rho_{j-2}}(0,0))$, Proposition 7.3.9 and relation (7.2.2) in Proposition 7.2.5 yield

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_2} \overline{u}_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} + (R/\rho_{j-2})^{-\frac{2}{q-1}},$$

which implies

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} + R^{-\frac{2}{q-1}}, \tag{7.4.5}$$

for some $\varepsilon = \varepsilon(N,q) > 0$.
First, we assume that there exists $J \in \mathbb{N}$, $J \geq 10$ such that

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \quad \forall\ j \geq J.$$

Then, from (7.4.1) and (7.4.2), (7.4.3), (7.4.4), (7.4.5), we have

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}} + 1,$$

for any $j \geq J$. Letting $R \to \infty$,

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + 1.$$

Since $\inf_{T_j} u \to \infty$ as $j \to \infty$, we get

$$\sum_{i=0}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

which implies that (7.1.14) holds with $(x,t) = (0,0)$.
Alternatively, assume that for infinitely many j

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} > \varepsilon$$

Then,

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N}} > \rho_{j-2}^{2-2q'}\varepsilon \to \infty \quad \text{when} \quad j \to \infty.$$

We also derive that (7.1.14) holds with $(x,t) = (0,0)$. This proves the case $q > q_*$.

Case 2 : $q = q_*$. Similarly to Case 1, we have : for $j \geq 6$

$$||u_4||_{L^\infty(T_j)} \lesssim 1, \tag{7.4.6}$$

$$||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=0}^{j-2} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{q-1}}, \tag{7.4.7}$$

$$||u_1||_{L^\infty(T_j)} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{7.4.8}$$

$$\mathrm{Cap}_{2,1,q'}(K_{j-5}) \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_{j-5})}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{7.4.9}$$

where $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x,t) \in K \cap Q_{\rho_{j-3}}(0,0)\}$ and $\varepsilon = \varepsilon(N) > 0$.
From (7.2.2) in Proposition 7.2.5, we have

$$\frac{1}{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\mathrm{Cap}_{2,1,q'}(K_j)} + cj^{N/2}$$

for any $j \geq 4$ where $c = c(N)$. If there are infinitely many $j \geq 4$ such that

$$\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) > \frac{1}{2cj^{N/2}},$$

then (7.1.14) holds with $(x,t) = (0,0)$ since

$$\frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^N} > \frac{2^{j-3}}{2cj^{N/2}} \to \infty \quad \text{when } j \to \infty.$$

Now, we assume that there exists $J \geq 6$ such that

$$\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2cj^{N/2}}.$$

Then,

$$\mathrm{Cap}_{2,1,q'}(K_j) \leq 2c\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \quad \forall\, j \geq J.$$

This leads to

$$\mathrm{Cap}_{2,1,q'}(K_j) \leq 2c\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall\, j \geq J' + J,$$

for some $J' = J'(N)$. Hence, from (7.4.6)-(7.4.9) we have, for any $j \geq J' + J + 3$,

$$\begin{aligned}
&||u_4||_{L^\infty(T_j)} \lesssim 1,\\
&||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=J'+J+1}^{j-2} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(J'+J) + jR^{-\frac{2}{q-1}},\\
&||u_1||_{L^\infty(T_j)} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}},\\
&\qquad \inf_{T_j} u_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-8}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}},
\end{aligned}$$

where $C(J'+J) = \sum_{i=0}^{J'+J} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_i^N}$.
Consequently we derive

$$\inf_{T_j} u \lesssim \sum_{i=0}^{j} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + C(J'+J) + 1 + jR^{-\frac{2}{q-1}} \quad \forall\, j \geq J' + J + 3$$

from (7.4.1). Letting $R \to \infty$ and $j \to \infty$ we obtain

$$\sum_{i=0}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

i.e (7.1.14) holds with $(x,t) = (0,0)$. This completes the proof of Theorem 7.1.1-(ii).

7.4.2 Proof of Theorem 7.1.1-(i) and Theorem 7.1.2

Fix $(x_0, t_0) \in \partial_p O$. We can assume that $(x_0, t_0) = 0$. Let $\delta \in (0, 1/100)$. For $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$, we set

$$M_k = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right)$$

and

$$S_k = \{(x,t) : r_{k+1} \leq \max\{|x - y_0|, |t - s_0|^{\frac{1}{2}}\} < r_k\} \text{ for } \ k = 1, 2, ...,$$

where $r_k = 4^{-k}$. Note that $M_k = \emptyset$ for k large enough and $M_k \subset S_k$ for all k. Let $R_0 \geq 4$ such that $O \subset\subset \tilde{Q}_{R_0}(0,0)$. By Theorem 7.2.2 and 7.2.4 and estimate (7.1.11) there exist two sequences $\{\mu_k\}_k$ and $\{\nu_k\}_k$ of nonnegative Radon measures such that

$$\mathrm{supp}(\mu_k) \subset M_k, \ \mathrm{supp}(\nu_k) \subset M_k, \tag{7.4.10}$$

$$\mu_k(M_k) \asymp \mathrm{Cap}_{2,1,q'}(M_k) \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^{2R_0}[\mu_k]\right)^q dxdt \tag{7.4.11}$$

and

$$\nu_k(M_k) \asymp \mathcal{PH}_1^N(M_k), \quad ||\mathbb{M}_1^{2R_0}[\nu_k]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{for } k = 1, 2, ..., \tag{7.4.12}$$

where the constants of equivalence depend on N, q, R_0.

Take $\varepsilon > 0$ such that $\exp\left(C_1\varepsilon\mathbb{I}_2^{2R_0}[\sum_{k=1}^\infty \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0,0))$ where the constant $C_1 = C_1(N)$ is the one of inequality (7.2.6). By Theorem 7.2.7 and Proposition 7.2.8, there exist two nonnegative solutions U_1, U_2 of problems

$$\begin{aligned} \partial_t U_1 - \Delta U_1 + U_1^q &= \varepsilon \sum_{k=1}^\infty \mu_k \quad && \text{in } \tilde{Q}_{R_0}(0,0), \\ U_1 &= 0 && \text{on } \partial_p\tilde{Q}_{R_0}(0,0). \end{aligned}$$

and

$$\begin{aligned} \partial_t U_2 - \Delta U_2 + e^{U_2} - 1 &= \varepsilon \sum_{k=1}^\infty \nu_k \quad && \text{in } \tilde{Q}_{R_0}(0,0), \\ U_2 &= 0 && \text{on } \partial_p\tilde{Q}_{R_0}(0,0), \end{aligned}$$

respectively which satisfy

$$\begin{aligned} U_1(y_0, z_0) \gtrsim & \sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon \frac{\mu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{35}{128}r_i^2))}{r_i^N} \\ & - \mathbb{I}_2^{2R_0}\left[\left(\mathbb{I}_2^{2R_0}[\varepsilon\sum_{k=1}^\infty \mu_k]\right)^q\right](y_0, s_0) =: A \end{aligned} \tag{7.4.13}$$

and

$$\begin{aligned} U_2(y_0, z_0) \gtrsim & \sum_{i=0}^\infty \sum_{k=1}^\infty \varepsilon \frac{\nu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{35}{128}r_i^2))}{r_i^N} \\ & - \mathbb{I}_2^{2R_0}\left[\exp\left(C_1\mathbb{I}_2^{2R_0}[\varepsilon\sum_{k=1}^\infty \nu_k]\right) - 1\right](y_0, s_0) =: B \end{aligned} \tag{7.4.14}$$

and $U_1, U_2 \in C^{2,1}(O)$.
Let u_1, u_2 be the maximal solutions of equations (7.3.1) and (7.3.16) respectively.
We have $u_1(y_0, s_0) \geq U_1(y_0, s_0)$ and $u_2(y_0, s_0) \geq U_2(y_0, s_0)$. Now, we claim that

$$A \gtrsim \sum_{k=1}^\infty \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N} \tag{7.4.15}$$

and

$$B \gtrsim -c_1(R_0) + \sum_{k=1}^\infty \frac{\mathcal{PH}_1^N(M_k)}{r_k^N}. \tag{7.4.16}$$

Proof of assertion (7.4.15)**.** From (7.4.11) we have

$$A \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1q'}(M_k)}{r_k^N} - \varepsilon^q A_0 \tag{7.4.17}$$

with

$$A_0 = \mathbb{I}_2^{2R_0}\left[\left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q\right](y_0, s_0).$$

Take $i_0 \in \mathbb{Z}$ such that $r_{i_0+1} < \max\{2R_0, 1\} \leq r_{i_0}$. Then

$$\begin{aligned}
A_0 &\lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{\tilde{Q}_{r_i}(y_0,s_0)} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&= \sum_{i=i_0}^{\infty}\sum_{j=i}^{\infty} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&= \sum_{j=k_0}^{\infty}\sum_{i=i_0}^{j} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt.
\end{aligned}$$

Here we have used the fact that $\sum_{i=i_0}^{j} r_i^{-N} \leq \frac{4}{3} r_j^{-N}$ for all j.
Setting $\mu_k \equiv 0$ for all $i_0 - 1 \leq k \leq 0$, the previous inequality becomes

$$\begin{aligned}
A_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\mu_j + \sum_{k=i_0-1}^{j-1}\mu_k + \sum_{k=j+1}^{\infty}\mu_k]\right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\mu_j]\right)^q dxdt \\
&+ \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} ||\mathbb{I}_2^{2R_0}[\mu_k]||_{L^\infty(S_j)}\right)^q \\
&+ \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=j+1}^{\infty} ||\mathbb{I}_2^{2R_0}[\mu_k]||_{L^\infty(S_j)}\right)^q \\
&= A_1 + A_2 + A_3.
\end{aligned} \tag{7.4.18}$$

Using (7.4.11) we obtain

$$A_1 \leq \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N}. \tag{7.4.19}$$

Next, using (7.4.10) we have for any $(x,t) \in S_j$ if $k \geq j+1$,

$$\mathbb{I}_2^{2R_0}[\mu_k](x,t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \tag{7.4.20}$$

and if $k \leq j-1$

$$\mathbb{I}_2^{2R_0}[\mu_k](x,t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}. \tag{7.4.21}$$

Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q$$

and

$$A_3 \lesssim \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q.$$

Noticing that $(a+b)^q - a^q \leq q(a+b)^{q-1}b$ for any $a, b \geq 0$, we get

$$\begin{aligned}
(1-4^{-2}) &\sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&= \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-2} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
(1-4^{2-Nq}) &\sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 + A_3 \lesssim &\sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N} \\
&+ \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Since $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q'}$ if $q > q_*$ and $\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q-1}}, 1\}$ if $q = q_*$ for any k, we infer that

$$r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \lesssim 1$$

and

$$r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \lesssim r_{j+1}^{-N} \quad \text{for any} \quad j.$$

In the case $q = q_*$ we assume $N \geq 3$ in order to ensure that

$$\sum_{j=1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q-1}} < \infty.$$

This leads to

$$A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}.$$

Combining this with (7.4.19) and (7.4.18), we deduce

$$A_0 \lesssim \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N}.$$

Consequently, we obtain (7.4.15) from (7.4.17), for ε small enough.

Proof of assertion (7.4.16)**.** From (7.4.12) we get

$$B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N} - B_0,$$

where

$$B_0 = \mathbb{I}_2^{2R_0} \left[\exp\left(C_1 \mathbb{I}_2^{2R_0}[\varepsilon \sum_{k=1}^{\infty} \nu_k] \right) - 1 \right] (y_0, s_0).$$

We show that

$$B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \ \text{ small enough.} \tag{7.4.22}$$

In fact, as above we have

$$B_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp\left(C_1 \varepsilon \mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty} \nu_k] \right) dxdt.$$

Consequently,

$$\begin{aligned}B_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp\Big(3C_1\varepsilon \mathbb{I}_2^{2R_0}[\nu_j]\Big)\, dxdt \\ &+ \sum_{j=i_0}^{\infty} r_j^2 \exp\left(3C_1\varepsilon \sum_{k=i_0-1}^{j-1} ||\mathbb{I}_2^{2R_0}[\nu_k]||_{L^\infty(S_j)}\right) \\ &+ \sum_{j=i_0}^{\infty} r_j^2 \exp\left(3C_1\varepsilon \sum_{k=j+1}^{\infty} ||\mathbb{I}_2^{2R_0}[\nu_k]||_{L^\infty(S_j)}\right) \\ &= B_1 + B_2 + B_3. \end{aligned} \tag{7.4.23}$$

Here we have used the inequality $\exp(a+b+c) \leq \exp(3a)+\exp(3b)+\exp(3c)$ for all a, b, c. By Theorem 7.2.3, we have

$$\int_{S_j} \exp\Big(3C_1\varepsilon \mathbb{I}_2^{2R_0}[\nu_j]\Big)\, dxdt \lesssim r_j^{N+2} \quad \text{for all } j,$$

for $\varepsilon > 0$ small enough. Hence,

$$B_1 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \tag{7.4.24}$$

Note that estimates (7.4.20) and (7.4.21) are also true with ν_k ; we deduce

$$\begin{aligned}B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_2\varepsilon \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}\right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_2\varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N}\right). \end{aligned}$$

From (7.4.12) we have $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N$ for all k, therefore

$$\begin{aligned}B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_3\varepsilon(j-i_0)\right) + \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_3\varepsilon\right) \\ &\lesssim \sum_{j=i_0}^{\infty} \exp\left(c_3\varepsilon(j-i_0) - 4\log(2)j\right) + r_{i_0}^2 \\ &\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \end{aligned}$$

Combining this with (7.4.24) and (7.4.23) we obtain (7.4.22).
This implies straightforwardly $\exp\Big(C_1\varepsilon \mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty} \nu_k]\Big) \in L^1(\tilde{Q}_{R_0}(0,0))$.
We conclude that for any $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$,

$$u_1(y_0, s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1q'}\left(M_k(y_0, s_0)\right)}{r_k^N}$$

and

$$u_2(y_0,s_0) \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N\left(M_k(y_0,s_0)\right)}{r_k^N},$$

where $r_k = 4^{-k}$ and

$$M_k(y_0,s_0) = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73+\frac{1}{2})r_{k+2}^2, s_0 - (70+\frac{1}{2})r_{k+2}^2]\right).$$

Take $r_{k_\delta+4} \le \delta < r_{k_\delta+3}$, we have for $1 \le k \le k_\delta$

$$\begin{aligned}
M_k(y_0,s_0) &\supset O^c \cap \left(B_{r_{k+2}-\delta}(0) \times \left(\delta^2 - (73+\frac{1}{2})r_{k+2}^2, -\delta^2 - (70+\frac{1}{2})r_{k+2}^2\right)\right)\\
&\supset O^c \cap \left(B_{r_{k+3}}(0) \times \left(-73r_{k+2}^2, -71r_{k+2}^2\right)\right)\\
&= O^c \cap \left(B_{r_{k+3}}(0) \times \left(-1168r_{k+3}^2, -1136r_{k+3}^2\right)\right).
\end{aligned}$$

Finally

$$\begin{aligned}
&\inf_{(y_0,s_0)\in(B_\delta(0)\times(-\delta^2,\delta^2))\cap O} u_1(y_0,s_0)\\
&\qquad \gtrsim \sum_{k=4}^{k_\delta+3} \frac{\mathrm{Cap}_{2,1,q'}\left(O^c \cap \left(B_{r_k}(0) \times \left(-1168r_k^2, -1136r_k^2\right)\right)\right)}{r_k^N} \to \infty \quad \text{as} \quad \delta \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{(y_0,s_0)\in(B_\delta(0)\times(-\delta^2,\delta^2))\cap O} u_2(y_0,s_0) \gtrsim -c_1(R_0)\\
&\qquad + \sum_{k=4}^{k_\delta+3} \frac{\mathcal{PH}_1^N\left(O^c \cap \left(B_{r_k}(0) \times \left(-1168r_k^2, -1136r_k^2\right)\right)\right)}{r_k^N} \to \infty \quad \text{as} \quad \delta \to 0.
\end{aligned}$$

This completes the proof of Theorem 7.1.1-(i) and Theorem 7.1.2.

7.4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation

$$\begin{aligned}
\partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 \qquad \text{in } O,\\
u &= \infty \qquad \text{on } \partial_p O,
\end{aligned} \tag{7.4.25}$$

where $a > 0, b > 0$ and $1 < p \le 2$, $q \ge 1$. First, we show that such a large solution to (7.4.25) does not exist when $q = 1$. Equivalently namely, for $a > 0$, $b > 0$ and $p > 1$ there exists no function $u \in C^{2,1}(O)$ satisfying

$$\begin{aligned}
\partial_t u - \Delta u + a|\nabla u|^p &\ge -bu \qquad \text{in } O,\\
u &= \infty \qquad \text{on } \partial_p O.
\end{aligned} \tag{7.4.26}$$

Indeed, assuming that such a function $u \in C^{2,1}(O)$, exists, we define

$$U(x,t) = u(x,t)e^{bt} - \frac{\varepsilon}{2}|x|^2,$$

for $\varepsilon > 0$ and denote by $(x_0, t_0) \in O \backslash \partial_p O$ the point where U achieves it minimum in O, i.e. $U(x_0,t_0) = \inf\{U(x,t) : (x,t) \in O\}$. Clearly, we have

$$\partial_t U(x_0,t_0) \leq 0, \quad \Delta U(x_0,t_0) \geq 0 \quad \text{and} \quad \nabla U(x_0,t_0) = 0.$$

Thus,

$$\partial_t u(x_0,t_0) \leq -bu(x_0,t_0), \quad -\Delta u(x_0,t_0) \leq -\varepsilon N e^{-bt_0} \quad \text{and} \quad a|\nabla u(x_0,t_0)|^p = a\varepsilon^p |x_0|^p e^{-pbt_0},$$

from which follows

$$\begin{aligned}\partial_t u(x_0,t_0) - \Delta u(x_0,t_0) + a|\nabla u(x_0,t_0)|^p &\leq -bu(x_0,t_0) + \varepsilon e^{-bt_0}\left(-N + a\varepsilon^{p-1}|x_0|^p e^{-(p-1)bt_0}\right) \\ &< -bu(x_0,t_0)\end{aligned}$$

for ε small enough, which is a contradiction.

Proof of Theorem 7.1.3. By Remark 7.3.3, we have

$$\inf\{v(x,t); (x,t) \in O\} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}.$$

Take $V = \lambda v^{\frac{1}{\alpha}} \in C^{2,1}(O)$ for $\lambda > 0$. Thus $v = \lambda^{-\alpha} V^{\alpha}$,

$$\inf\{V(x,t); (x,t) \in O\} > 0\} \geq \lambda (q_1 - 1)^{-\frac{1}{\alpha(q_1-1)}} R^{-\frac{2}{\alpha(q_1-1)}},$$

and

$$\partial_t v - \Delta v + v^{q_1} = \alpha\lambda^{-\alpha}V^{\alpha-1}\partial_t V - \alpha\lambda^{-\alpha}V^{\alpha-1}\Delta V + \alpha(1-\alpha)\lambda^{-\alpha}V^{\alpha-1}\frac{|\nabla V|^2}{V} + \lambda^{-\alpha q_1}V^{\alpha q_1}.$$

This leads to

$$\partial_t V - \Delta V + (1-\alpha)\frac{|\nabla V|^2}{V} + \alpha^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1 - \alpha + 1} = 0 \quad \text{in} \quad O.$$

Using Hölder's inequality,

$$\begin{aligned}(1-\alpha)\frac{|\nabla V|^2}{V} + (2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1 - \alpha + 1} &\geq c_1 |\nabla V|^p \lambda^{-\frac{\alpha(q_1-1)(2-p)}{2}} V^{\frac{\alpha(q_1-1)(2-p)}{2} - (p-1)} \\ &\geq c_2 |\nabla V|^p \lambda^{-(p-1)} R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}}\end{aligned}$$

and

$$(2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1 - \alpha + 1} \geq c_3 \lambda^{-(q-1)} R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} V^q.$$

If we choose

$$\lambda = \min\{c_2^{\frac{1}{p-1}}, c_3^{\frac{1}{q-1}}\} \min\left\{a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}}\right\}$$

then

$$c_2\lambda^{-(p-1)}R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} \geq a,$$
$$c_3\lambda^{-(q-1)}R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} \geq b,$$

from what follows

$$\partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in} \;\; O.$$

By Remark 7.3.5, there exists a maximal solution $u \in C^{2,1}(O)$ of

$$\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in} \;\; O.$$

Therefore, $u \geq V = \lambda v^{\frac{1}{\alpha}}$ and u is a large solution of (7.4.25). This completes the proof of Theorem 7.1.3. ∎

7.5 Appendix

Proof of Proposition 7.2.5.

Step 1. We claim that the following relation holds :

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^1[\mu](x,t)\right)^{(N+2)/N} dxdt \asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t). \tag{7.5.1}$$

In fact, we have for $\rho_j = 2^{-j}$, $j \in \mathbb{Z}$,

$$\begin{aligned}\sum_{j=1}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N} d\mu(x,t) &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) \\ &\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N} d\mu(x,t).\end{aligned}$$

Note that for any $j \in \mathbb{Z}$

$$\begin{aligned}\rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j+1}}(x,t)))^{(N+2)/N} dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N} d\mu(x,t) \\ &\lesssim \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j-1}}(x,t)))^{(N+2)/N} dxdt.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{j=2}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N} dxdt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) \\ &\lesssim \sum_{j=-1}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N} dxdt.\end{aligned}$$

This yields

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} \left(\mathbb{M}_2^{1/4}[\mu](x,t)\right)^{(N+2)/N} dxdt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) \\ &\lesssim \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^4[\mu](x,t)\right)^{(N+2)/N} dxdt.\end{aligned}$$

By [20, Theorem 4.2],

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{M}_2^{1/4}[\mu](x,t)\right)^{(N+2)/N} dxdt \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^4[\mu](x,t)\right)^{(N+2)/N} dxdt,$$

thus we obtain (7.5.1).

Step 2. End of the proof. The first inequality in (7.2.1) is proved in [20]. We now prove the second inequality. By Theorem 7.2.4 there is $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), \mathrm{supp}(\mu) \subset K$ such that

$$||\mathbb{M}_2^2[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \text{ and } \mu(K) \asymp \mathcal{PH}_2^N(K) \gtrsim |K|^{N/(N+2)}. \tag{7.5.2}$$

Thanks to (7.5.1), we have for $\delta = \min\{1, (\mu(K))^{1/N}\}$

$$\begin{aligned}||\mathbb{I}_2^1[\mu]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} &\asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) \\ &\asymp \int_{\mathbb{R}^{N+1}} \left(\int_0^\delta + \int_\delta^1\right) (\mu(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) \\ &\lesssim \int_0^\delta r^2 \frac{dr}{r} \int_{\mathbb{R}^{N+1}} d\mu(x,t) + \int_\delta^1 \frac{dr}{r} \left(\int_{\mathbb{R}^{N+1}} d\mu(x,t)\right)^{(N+2)/N} \\ &\lesssim (\mu(K))^{(N+2)/N} \left(1 + \log_+ \left((\mu(K))^{-1}\right)\right) \\ &\lesssim (\mu(K))^{(N+2)/N} \log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right).\end{aligned}$$

Set $\tilde{\mu} = \left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right)\right)^{-N/(N+2)} \mu/\mu(K)$, then $||\mathbb{I}_2^1[\tilde{\mu}]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1$.
It is well known that

$$\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K) \asymp \sup\{(\omega(K))^{(N+2)/2} : \omega \in \mathfrak{M}^+(K), ||\mathbb{I}_2^1[\omega]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\} \tag{7.5.3}$$

see [20, Section 4]. This gives the second inequality in (7.2.1).
It is easy to prove (7.2.2) from its definition. Moreover, (7.5.3) implies that

$$\frac{1}{\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K)^{2/N}} \asymp \inf\{||\mathbb{I}_2^1[\omega]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\}.$$

We deduce from (7.5.1) that

$$\frac{1}{\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K)^{2/N}} \asymp \inf\left\{\int_{\mathbb{R}^{N+1}} \int_0^1 (\omega(\tilde{Q}_r(x,t)))^{2/N} \frac{dr}{r} d\mu(x,t) : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\right\}. \tag{7.5.4}$$

As in [12, proof of Lemma 2.2], it is easy to derive (7.2.3) from (7.5.4). ■

Proof of Proposition 7.2.6. Thanks to the Poincaré inequality, it is enough to show that there exists $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))$ such that $0 \le \varphi \le 1$, with $\varphi = 1$ in an open neighborhood of K and

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t \varphi|^p) dxdt \lesssim \mathrm{Cap}_{2,1,p}(K). \tag{7.5.5}$$

By definition, one can find $0 \le \phi \in S(\mathbb{R}^{N+1})$, $\phi \ge 1$ in a neighborhood of K such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \le 2\mathrm{Cap}_{2,1,p}(K).$$

Let η be a cut off function on $\tilde{Q}_1(0,0)$ with respect to $\tilde{Q}_{3/2}(0,0)$ and $H \in C^\infty(\mathbb{R})$ such that

$0 \le H(t) \le t^+,\ |t||H''(t)| \lesssim 1$ for all $t \in \mathbb{R},\ H(t) = 0$ for $t \le 1/4$ and $H(t) = 1$for $t \ge 3/4$.

We claim that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt, \tag{7.5.6}$$

where $\varphi = \eta H(\phi)$. Indeed, we have

$$|D^2\varphi| \lesssim |D^2\eta| H(\phi) + |\nabla\eta||H'(\phi)||\nabla\phi| + \eta|H''(\phi)||\nabla\phi|^2 + \eta|H'(\phi)||D^2\phi|$$

and

$$|\partial_t\varphi| \lesssim |\partial_t\eta| H(\phi) + \eta|H'(\phi)||\phi_t|, \quad H(\phi) \le \phi, \quad \phi|H''(\phi)| \lesssim 1.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \\ &\quad + \int_{\mathbb{R}^{N+1}} \frac{|\nabla\phi|^{2p}}{\phi^p} dxdt.\end{aligned}$$

This implies (7.5.6) since, according to [1], one has

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi(t)|^{2p}}{\phi(t)^p} dx \lesssim \int_{\mathbb{R}^N} |D^2\phi(t)|^p dx \quad \forall t \in \mathbb{R}.$$

■

Bibliographie

[1] D. R. Adams. *On the existence of capacitary strong type estimates in* $\mathbb{R}^N$, Ark. Mat. **14**, 125-140 (1976).

[2] R.J. Bagby. *Lebesgue spaces of parabolic potentials*, Ill. J. Math. **15**, 610-634 (1971).

[3] P. Baras and M. Pierre. *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[4] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, (1985), 185-212.

[5] J. S. Dhersin and J. F. Le Gall. *WienerâĂŹs test for super-Brownian motion and the Brownian snake*, Probab. Theory Relat. Fields **108**, 103-29 (1997).

[6] E. DiBenedetto. *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, (1993).

[7] E. B. Dynkin, S. E. Kuznetsov, *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations.* Comm. Pure Appl. Math. **51**, 897-936 (1998).

[8] E. B. Dynkin, S. E. Kuznetsov, *Superdiffusions and removable singularities for quasi-linear partial differential equations.* Comm. Pure Appl. Math. **49**, 125-176 (1996).

[9] L. C. Evans, R. F. Gariepy, *Wiener criterion for the heat equation.* Arch. Rat. Mech. Anal. **78**, 293-314 (1982).

[10] N. Garofalo and E. Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans. Amer. Math. Soc. **308**, No.2, 811-836 (1988).

[11] R. F. Gariepy and W.P. Ziemer, *Thermal capacity and boundary regularity*, J. Diff. Eq. **45**, 374-388 (1982).

[12] D. Labutin. *Wiener regularity for large solutions of nonlinear equations*, Archiv for Math. **41**, 307-339 (2003).

[13] J.-F. LeGall, *Branching processes, random trees and superprocesses.* Proceedings of the International Congress of Mathematicians (Berlin, 1998), Doc. Math., Extra Vol. III 279-289 (1998).

[14] J.-F. LeGall, *Spatial branching processes, random snakes and partial differential equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[15] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific press, River Edge (1996).

[16] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, Series in Nonlinear Analysis and Applications **21**, De Gruyter, Berlin/Boston (2013).

[17] M. Marcus and L. Véron. *Capacitary estimates of solutions of semilinear parabolic equations*, Calc. Var. & Part. Diff. Equ. **48**, 131-183 (2013).

[18] T. Nguyen-Phuoc *Parabolic equations with exponential nonlinearity and measure data*, J. Diff. Equ. (to appear), arXiv :1312.2509.

[19] H. Nguyen-Quoc and L. Véron., *Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption*, arXiv :1308.2956v3.

[20] H. Nguyen-Quoc, *Potential estimates and quasilinear equations with measure data* arXiv :1405.2587v1.

[21] B. O. Tureson. *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, Vol. 1736, Springer-Verlag (2000).

[22] L. Véron. *A note on maximal solutions of nonlinear parabolic equations with absorption*, Asymptotic Anal. **72**, 189-200 (2011).

[23] W.P. Ziemer, *Behavior at the boundary of solution of quasilinear parabolic equations*, J. Diff. Eq. **35**, 291-305 (1980).